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MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

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p. 40, Question 3123, part 2; the value of the mean here should be $\frac{3}{4}$B, as given in the Editor's Note on p. 72 of Vol. XX. of the *Reprint*.

VOL. XIX.

pp. viii. and 17, *for* Question 3229 *read* Question 3929.

VOL. XX.

p. 26, equation (18), *for* $\frac{1}{r} - \frac{d}{dt}\left(r^2\frac{d\vartheta}{dt}\right)$ *read* $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\vartheta}{dt}\right)$;

p. 41, line 17, *dele* $\frac{AQ'.BQ'}{AP'.BP'}$;

" line 18, *for* $\frac{QN}{PN}$ *read* $\frac{QN'}{PN}$, and refer to the figure given on p. 68;

p. 47, line 10, *for* $\mu\frac{r^2}{s^2}$ *read* μr^2;

" line 11, *for* $\mu\frac{R^2}{s^2}$ *read* μR^2.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

4092. (Proposed by J. J. SYLVESTER, F.R.S.)—1. If a body move in an ellipse with *uniform* velocity acted on by forces to the foci, prove that they are equal, and vary inversely as the square of the conjugate diameter.

2. If a body describe any orbit, acted on by force f, f', tending to two fixed points, and P, P′ be the squared perpendiculars drawn from the fixed points upon the tangent, prove that

$$f = \frac{d}{dr}\left(\frac{\int \phi\, d\mathrm{P}}{\mathrm{P}}\right),\quad f' = -\frac{d}{dr'}\left(\frac{\int \phi\, d\mathrm{P}'}{\mathrm{P}'}\right),$$

ϕ being an arbitrary affection of the position of the body in its orbit; and extend the result to any number of centres of force.

Solution by Professor WOLSTENHOLME, M.A.

1. Since the velocity u is uniform, the resultant force must be normal and vary as the curvature, hence if F, F′ be the two forces and 2ϕ the angle between the focal distances, $\mathrm{F}\sin\phi = \mathrm{F}'\sin\phi$, or $\mathrm{F} = \mathrm{F}'$ and $\mathrm{F}\cos\phi + \mathrm{F}'\cos\phi = \dfrac{u^2}{\rho}$ or $2\mathrm{F}\cos\phi = \dfrac{u^2}{\rho}$; therefore $\mathrm{F} = u^2 \div$ chord of curvature through the focus $= \dfrac{u^2 . \mathrm{AC}}{2\mathrm{CD}^2}$, with the usual notation.

The same proposition is true if the lines of action of the forces are tangents to any fixed confocal ellipse.

2. $$\frac{d^2 s}{dt^2} = v\frac{dv}{ds} = -f\frac{dr}{ds} - f'\frac{dr'}{ds},\ \text{and}\ \frac{v^2}{\rho} = f\frac{p}{r} + f'\frac{p'}{r'},$$

or $$v^2 = fp\frac{dr}{dp} + f'p'\frac{dr'}{dp'};$$

therefore $$fp\frac{dr}{dp} + f'p'\frac{dr'}{dp'} = -2\int f\,dr - 2\int f'\,dr',$$

or $$fp\frac{dr}{dp} + 2\int f\,dr = -\left(f'p'\frac{dr'}{dp'} + 2\int f'\,dr'\right),$$

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and if we put each of these equal to 2ϕ any arbitrary function of the position of the point we have

$$\frac{1}{p}\frac{d}{dp}\left(p^2\int f\,dr\right) = 2\phi, \text{ or } p^2\int f\,dr = \int 2p\phi\,dp;$$

or
$$\int f\,dr = \frac{1}{\text{P}}\int \phi\,d\text{P}, \text{ or } f = \frac{d}{dr}\left(\frac{1}{\text{P}}\int \phi\,d\text{P}\right),$$

and similarly
$$f' = -\frac{d}{dr'}\left(\frac{1}{\text{P}'}\int \phi\,d\text{P}'\right).$$

4037. (Proposed by Prof. Townsend, F.R.S.)—A hollow cylinder, of uniform thickness, material, and elasticities tangential and radial, being supposed pressed from within and from without by radial forces of uniform intensities per unit of area; if α, β, γ be the three radial extensions, and A, B, C the three radial pressures per unit of area, at any three distances a, b, c from the axis, prove the relations

$$a^k(b^{2k}-c^{2k})\,\alpha + b^k(c^{2k}-a^{2k})\,\beta + c^k(a^{2k}-b^{2k})\,\gamma = 0,$$
$$a^k(b^{2k}-c^{2k})\,\text{A}a + b^k(c^{2k}-a^{2k})\,\text{B}b + c^k(a^{2k}-b^{2k})\,\text{C}c = 0;$$

where k^2 = the ratio of the modulus E of tangential to the modulus E′ of radial elasticity, each of which within the limits of actual displacement is supposed to be the same for extension and compression.

Solution by the Proposer.

For, denoting by ρ the radial extension, and by R the radial pressure per unit of area at any distance r from the axis, then, from the differential equations of equilibrium of the cylindrical strata, viz.,

$$d(\text{R}r) + \text{E}\frac{\rho}{r}\,dr = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

for tangential, and
$$\text{R} + \text{E}'\frac{d\rho}{dr} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

for radial equilibrium, since, by multiplication,

$$(\text{R}r)\,.\,d(\text{R}r) = \text{EE}'\,.\,\rho\,d\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)},$$

and therefore, by integration, for any distance r,

$$(\text{R}r)^2 = \text{EE}'\,.\,(\rho^2 \pm \delta^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(4)};$$

and since, consequently, from (2) and (4),

$$\frac{d\rho}{(\rho^2 \pm \delta^2)^{\frac{1}{2}}} = -\left(\frac{\text{E}}{\text{E}'}\right)^{\frac{1}{2}}\cdot\frac{dr}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(5)},$$

and therefore again, by integration, for any two distances r and r',

$$\frac{\rho \pm (\rho^2 \pm \delta^2)^{\frac{1}{2}}}{\rho' \pm (\rho'^2 \pm \delta^2)^{\frac{1}{2}}} = \frac{\rho \pm \dfrac{\text{R}r}{(\text{EE}')^{\frac{1}{2}}}}{\rho' \pm \dfrac{\text{R}'r'}{(\text{EE}')^{\frac{1}{2}}}} = \left(\frac{r'}{r}\right)^{\pm\left(\frac{\text{E}}{\text{E}'}\right)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots \text{(6)};$$

therefore, for any three distances a, b, c,

$$a^{\left(\frac{E}{E'}\right)^{\frac{1}{2}}}\left(\alpha+\frac{Aa}{(EE')^{\frac{1}{2}}}\right)=b^{\left(\frac{E}{E'}\right)^{\frac{1}{2}}}\left(\beta+\frac{Bb}{(EE')^{\frac{1}{2}}}\right)=c^{\left(\frac{E}{E'}\right)^{\frac{1}{2}}}\left(\gamma+\frac{Cc}{(EE')^{\frac{1}{2}}}\right),$$

$$a^{-\left(\frac{E}{E'}\right)^{\frac{1}{2}}}\left(\alpha-\frac{Aa}{(EE')^{\frac{1}{2}}}\right)=b^{-\left(\frac{E}{E'}\right)^{\frac{1}{2}}}\left(\beta-\frac{Bb}{(EE')^{\frac{1}{2}}}\right)=c^{-\left(\frac{E}{E'}\right)^{\frac{1}{2}}}\left(\gamma-\frac{Cc}{(EE')^{\frac{1}{2}}}\right),$$

and therefore, &c., the relations above following, of course, immediately from these last.

COROLLARY.—If a and b be the inner and outer radii of the cylinder, and if the pressure B at the outer surface $=0$, it follows at once from the above that

$$\alpha=\frac{b^{2\left(\frac{E}{E'}\right)^{\frac{1}{2}}}+a^{2\left(\frac{E}{E'}\right)^{\frac{1}{2}}}}{b^{2\left(\frac{E}{E'}\right)^{\frac{1}{2}}}-a^{2\left(\frac{E}{E'}\right)^{\frac{1}{2}}}}\cdot\frac{Aa}{(EE')^{\frac{1}{2}}},$$

the least value of which, where b is indefinitely increased, $=\dfrac{Aa}{(EE')^{\frac{1}{2}}}$. Which (as shown by Dr. Hart by a process precisely similar to the above) gives the limit to the value of A when the extreme value of α is consistent with the safety of the cylinder is known by experiment.

4071. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—A cone whose vertex is on a surface of the second order envelopes a confocal surface; find the length of the axis of the cone between the vertex and the plane of contact.

Solution by R. F. SCOTT.

Let the equations of the quadrics on which the point lies, and to which the tangent cone is drawn, be

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\text{.........}\quad (1, 2).$$

Let also (1) be that quadric to which the axis we consider is normal.

The equations of the axis are

$$\frac{\xi-x}{\dfrac{x}{a^2+\lambda}}=\frac{\eta-y}{\dfrac{y}{b^2+\lambda}}=\frac{\zeta-z}{\dfrac{z}{c^2+\lambda}}=-Rp \quad\text{.....................}\quad (3),$$

where R is the distance between the points (ξ,η,ζ) and (x,y,z), and p is the perpendicular on the tangent plane to (1).

The polar plane of (x,y,z), with regard to (2), is

$$\frac{\xi x}{a^2}+\frac{\eta y}{b^2}+\frac{\zeta z}{c^2}=1 \quad\text{..................................}\quad (4),$$

where (3) and (4) meet, we readily obtain

$$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right\}\left\{\frac{Rp}{\lambda}-1\right\}=0,\quad\text{therefore } R=\frac{\lambda}{p}.$$

Whence we get the values of the intercepts made by the three axes.

4102. (Proposed by T. T. Wilkinson, F.R.A.S.)—Prove the following Rule for finding right-angled triangles whose legs differ by a given number:—Write the sides of the original triangle under each other with the hypotenuse second. Then make a second column thus: take the sum of the two upper figures for a new top figure; the sum of the two lower for a new bottom figure; and the sum of all three for a new middle figure. Repeat the operation on this second column so as to get a third, and this column will represent the sides of a new triangle fulfilling the conditions. Continuing the process to any number of columns, the alternate ones, namely, each that contains two odd numbers, will give a prime right-angled triangle with the same difference of legs as the first. The following are several such triangles:—

	I		II		III		IV	V	VI	
Base	3	8	20	49	119	288	696	4059	23660	
Hyp.	5	12	29	70	169	408	985	5741	33461	&c.
Perp.	4	9	21	50	120	289	697	4060	23661	

I. *Solution by* Henry Stanley Monck.

Let the figures in the first row be a, b, c; then, since $b^2 = a^2 + c^2$, we may put $a = b \sin\theta$, $c = b\cos\theta$, and the rows will then stand as follows:—

$$b\begin{cases}\sin\theta, & a, \\ 1\ \ , & b, \\ \cos\theta, & c.\end{cases} \quad b\begin{cases}1+\sin\theta, & a_2, \\ 1+\sin\theta+\cos\theta, & b_2, \\ 1+\cos\theta, & c_2.\end{cases} \quad b\begin{cases}2+2\sin\theta+\ \ \cos\theta, & a_3, \\ 3+2\sin\theta+2\cos\theta, & b_3, \\ 2+\ \ \sin\theta+2\cos\theta, & c_3.\end{cases}$$

The difference of the first and third members of each row is evidently $b(\sin\theta - \cos\theta)$. It remains to show that the square of the second member is equal to the sum of the squares of the other two. Now we have

$$b^2_3 - a^2_3 = b^2(5+4\sin\theta+3\cos\theta)(1+\cos\theta).$$

But
$$\begin{aligned} c^2_3 &= b^2(2+2\cos\theta+\sin\theta)^2 \\ &= b^2\left\{(2+2\cos\theta)^2+2(2+2\cos\theta)\sin\theta+\sin^2\theta\right\} \\ &= b^2(1+\cos\theta)(5+4\sin\theta+3\cos\theta) = b^2_3 - a^2_3. \end{aligned}$$

Hence the third column satisfies all the conditions of the first, and if we make $a_3 = b_3 \sin\phi$, $c_3 = b_3\cos\phi$, we can prove the same thing with respect to the fifth row, and so on, as long as we please.

It is easy to show that in the second (and by consequence in all the even) columns, the square of the second member is greater than the sum of the squares of the first and third. The excess, in fact, is $b(1-\sin 2\theta)$, which cannot vanish under the conditions of the problem.

The table in the question contains implicitly a solution of the problem to find the whole number of pairs of integers, the sum of whose squares is itself a square. I shall, in proving this, first show that the table contains a complete list of the numbers whose difference is 1, and whose sum is a square. Using the above designations, we find

$$a_3 = 2a+2b+c, \quad b_3 = 2a+3b+2c, \quad c_3 = a+2b+2c.$$

Proceeding now to express a, b, c in terms of a_3, b_3, c_3 we get

$$a = 2a_3+c_3-2b_3, \quad b = 3b_3-2a_3-2c_3, \quad c = 2c_3+a_3-2b_3.$$

It is easy to see that if a_3, b_3, c_3 are the three sides of a right-angled triangle, $a, b,$ and c will equally form three sides of another (*i.e.*, that $a^2+c^2 = b^2$) and that $a_3 - c_3 = a - b$. Further, a, b, c will be integers if

a_3, b_3, c_3 are so, and they will be less than a_3, b_3, c_3 respectively, as will appear most easily from the inverse expression. This shows that the process in the question can be inverted, and three numbers, all less than the three we start from (provided these are positive integers), can be found fulfilling the same conditions.

Now let A, B, C be any three numbers, such that $A-C=1$ (and the same reasoning will apply if $A-C=m$) and $A^2+C^2=B^2$. We can, as above, obtain three smaller numbers fulfilling the same conditions; and performing the same process on these latter, we can obtain three others still smaller, and so on backwards till we reach the limit. This limit is reached when $C_n=0$, when (the difference being 1) A_n will be $=1$, and $B_n=1$ also. Hence all integers, whose difference is 1 and the sum of whose squares is a square, may be found by operating by the method in the question on the column $\begin{matrix}1\\1\\0\end{matrix}$ since the inverse operation always leads back to it. Here the second column is $\begin{matrix}2\\2,\\1\end{matrix}$ and the third $\begin{matrix}4\\5,\\3\end{matrix}$ which is the first in the question, and therefore the table gives all the required numbers except 1 and 0, which plainly fulfil the conditions, since $1-0=1$ and $1^2+0^2=1^2$. If we attempt to go back farther here, we only get 1, 1 and 0 over again.

Let us now, instead of 1, take m for our difference, and let α, β, γ be three numbers, such that $\alpha-\gamma=m$, $\alpha^2+\gamma^2=\beta^2$. Proceeding in the inverse method, we get smaller and smaller numbers fulfilling the same conditions till we reach the limit which plainly is $\alpha_n=m$, $\beta_n=m$, $\gamma_n=0$, where again, if we try to go back farther, we get the same values for $\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}$ and all subsequent numbers. Hence all numbers whose difference is m, and the sum of whose squares is a square, are found by operating on the column $\begin{matrix}m\\m,\\0\end{matrix}$ since, pursuing the inverse process, we always get back to it.

Let us compare the operation with that on the column $\begin{matrix}1\\1\\0\end{matrix}$

I		II		III		IV		V	
1	2	4	9	21	50	120	289	697	1672
1	2	5	12	29	70	169	408	985	2368
0	1	3	8	20	49	119	288	696	1671
m	$2m$	$4m$	$9m$	$21m$	$50m$				
m	$2m$	$5m$	$12m$	$29m$	$70m$	&c.			
0	m	$3m$	$8m$	$20m$	$49m$				

Here it is evident that all the numbers fulfilling the latter conditions are those which fulfil the former multiplied by m; and therefore *all pairs of integers, the sum of whose squares is a perfect square, are to be found in the table and its integer multiples.* I am not aware that any mode of setting out a complete list of such numbers has yet been recognized.

To show how Question 4177 follows from the theorem given, I may make this remark. As the first and third numbers in the table become larger at each successive step, while their difference remains constant, their ratio increases. Hence $\frac{3}{4}$ and $\frac{3}{5}$ are the least values of $\frac{c}{a}$ and $\frac{b}{a}$ respectively.

II. *Solution by the* Rev. G. H. HOPKINS, M.A.

If x, $x+1$, y satisfy the equation $\alpha^2+\beta^2 = \gamma^2$,

then $$2x^2+2x+1 = y^2 \text{ or } (2x+1)^2-(y\sqrt{2})^2 = -1.$$

Now $x=0$, and $y=1$ satisfies this equation, hence by Todhunter's *Algebra*, Art. 640, we have

$$x = \frac{1}{4}\left\{(1+\sqrt{2})^n+(1-\sqrt{2})^n\right\}-\frac{1}{2},$$

$$y = \frac{1}{2\sqrt{2}}\left\{(1+\sqrt{2})^n-(1-\sqrt{2})^n\right\},$$

$$x+1 = \frac{1}{4}\left\{(1+\sqrt{2})^n+(1-\sqrt{2})^n\right\}+\frac{1}{2},$$

n being an odd integer.

According to the rule $3x+2y+1$, $4x+3y+2$, and $3x+2y+2$ are the third set of values which satisfy the equation; substituting the values of x and y as obtained, we find

$$x' = \frac{3}{4}\left\{(1+\sqrt{2})^n+(1-\sqrt{2})^n\right\}-\frac{3}{2}+\frac{2}{2\sqrt{2}}\left\{(1+\sqrt{2})^n-(1-\sqrt{2})^n\right\}+1$$

$$= \frac{1}{4}\left\{(1+\sqrt{2})^n(3+2\sqrt{2})+(1-\sqrt{2})^n(3-2\sqrt{2})\right\}-\frac{1}{2}$$

$$= \frac{1}{4}\left\{(1+\sqrt{2})^{n+2}+(1-\sqrt{2})^{n+2}\right\}-\frac{1}{2},$$

of the same form as the above.

Similarly, $$y' = 4x+3y+2 = \frac{1}{2\sqrt{2}}\left\{(1+\sqrt{2})^{n+2}-(1-\sqrt{2})^{n+2}\right\}.$$

4074. (Proposed by CHRISTINE LADD.)—Let ABCD be a quadrilateral (P), and O its centre of inertia. Form a quadrilateral (P′), the sides of which are equal in length and direction to OA, OB, OC, OD. Show that lines joining middle points of opposite sides of (P) are equal in length and direction to diagonals of (P′), and that diagonals of (P) are equal in length and direction to twice lines joining middle points of opposite sides of (P′).

Solution by R. F. SCOTT; *the* PROPOSER; *and others.*

Let the vertices A, B, C, D be given by the vectors α, β, γ, δ, and let t be the vector of O; so that $4t = \alpha+\beta+\gamma+\delta$.

Then, if A′, B′, C′, D′ be the new quadrilateral, we shall have

$$A'B' = \alpha-t,\quad A'C' = \alpha+\beta-2t,\quad A'D' = \alpha+\beta+\gamma-3t.$$

Now $$A'C' = \tfrac{1}{2}(\alpha+\beta)-\tfrac{1}{2}(\gamma+\delta),$$

$$B'D' = A'D'-A'B' = -(\delta-t)-(\alpha-t) = \tfrac{1}{2}(\beta+\gamma)-\tfrac{1}{2}(\alpha+\delta).$$

That is to say, the diagonals of P′ are equal in magnitude to the lines joining the middle points of opposite sides of P, and are parallel to them.

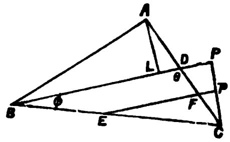
The line joining the middle points of A′B′, C′D′ is

$$\tfrac{1}{2}(\alpha+\beta-2t+\alpha+\beta+\gamma-3t)-\tfrac{1}{2}(\alpha-t) = \tfrac{1}{2}(\beta-\delta) = \tfrac{1}{2}(\text{BD}),$$

which proves the second part.

3693. (Proposed by S. WATSON.)—A line is drawn at random so as to cut a given triangle, and two points are taken at random within the triangle; find the chance that the points lie on opposite sides of the line.

Solution by G. S. CARR.



Let EF be a random line making an angle θ with AC; and first let θ lie between A and A + B. Draw BD parallel to EF, and AL and CP perpendicular to BD. Let $Cp = x$.

The probability of the line crossing the triangle at a given inclination θ and at a given distance x from C is

$$\frac{dx}{\mathrm{AL}+\mathrm{CP}}\cdot\frac{d\theta}{\pi}=\frac{dx\,d\theta}{\pi b\sin\theta}\ \ldots\ldots\ldots\ldots\ (1).$$

The subsequent probability of the two points being found on opposite sides of EF is

$$\frac{2\mathrm{CEF}}{\mathrm{ABC}}\left\{1-\frac{\mathrm{CEF}}{\mathrm{ABC}}\right\}\ \ldots\ldots\ldots\ldots\ (2).$$

But $\dfrac{\mathrm{CEF}}{\mathrm{ABC}}=\dfrac{\mathrm{CEF}}{\mathrm{DBC}}\cdot\dfrac{\mathrm{DBC}}{\mathrm{ABC}}=\dfrac{x^2}{\mathrm{BP}^2}\cdot\dfrac{\mathrm{BP}}{\mathrm{AL}+\mathrm{BP}}=\dfrac{x^2}{ab\sin(\theta+\mathrm{C})\sin\theta}.$

Substituting this in (2) and multiplying by (1) we get for the probability of the required event happening with the given values of x and θ

$$2\frac{d\theta\,dx}{\pi}\left\{\frac{x^2}{ab^2\sin^2\theta\sin(\theta+\mathrm{C})}-\frac{x^4}{a^2b^3\sin^3\theta\sin^2(\theta+\mathrm{C})}\right\}\ \ldots\ldots\ (3).$$

Integrating this result between the limits $x=0$ and $x=a\sin(\theta+\mathrm{C})$ we obtain for the probability when θ alone is given and the line crosses between C and D

$$\left\{\frac{2a^2}{3b^2}\cdot\frac{\sin^2(\theta+\mathrm{C})}{\sin^2\theta}-\frac{2a^3}{5b^3}\cdot\frac{\sin^3(\theta+\mathrm{C})}{\sin^3\theta}\right\}\frac{d\theta}{\pi}\ \ldots\ldots\ldots\ (4).$$

If EF crosses between D and A, we find the remaining part of the probability for this inclination in a precisely similar way by measuring x from A to L, viz.,

$$\left\{\frac{2c^2}{3b^2}\cdot\frac{\sin^2(\theta-\mathrm{A})}{\sin^2\theta}-\frac{2c^3}{5b^3}\cdot\frac{\sin^3(\theta-\mathrm{A})}{\sin^3\theta}\right\}\frac{d\theta}{\pi}\ \ldots\ldots\ldots\ (5).$$

The sum of these two expressions must be integrated with respect to θ between the limits A and A + B. The integrals involved are

$$\int\cot\theta\,d\theta=\log\sin\theta,\quad\int\cot^2 d\theta=-\cot\theta-\theta\ \text{and}\int d\theta=\theta;$$

$\int\cot^3\theta\,d\theta$ also appears, but is eliminated without integration. The result is

$$\frac{1}{\pi}\left[\left\{\tfrac{2}{15}(a^2+c^2)-\tfrac{1}{5}a^2\sin^2\mathrm{C}+\tfrac{1}{10}ac\cos\mathrm{A}\cos\mathrm{C}\right\}\frac{2\mathrm{B}}{b^2}+\frac{2ac}{15b^2}\sin(\mathrm{A}-\mathrm{C})\log\frac{c}{a}+\frac{2a}{15b}\sin\mathrm{C}\right].$$

This together with two similar expressions written out symmetrically for the remaining angles of the triangle constitute the whole probability required. When the triangle is equilateral the probability reduces to

$$\frac{2}{15}+\frac{\sqrt{3}}{5\pi}=\cdot 24359909.$$

NOTE.—The foregoing solution does not agree with that given by the

Proposer (on p. 47 of Vol. XVIII. of the *Reprint*), and I am compelled to differ from him as to the principle of solution adopted. I consider that the total probability of the two points being found on opposite sides of the line must be equal to the sum of the separate probabilities for each successive inclination of the line. One of these separate probabilities is represented by $(4)+(5)$ in the above solution, and these expressions are found in (1) and (2) of Mr. WATSON's Solution with the divisor $b \sin \theta = b \sin (C + \phi)$ omitted. Let $(4)+(5)$ be denoted by $\frac{F(\theta)}{b \sin \theta} \cdot \frac{d\theta}{\pi}$; then the distinction between the two modes of solution is this :—

Mr. WATSON and I consider the whole probability to be, respectively,

$$\frac{\int F(\theta)\, d\theta}{\int b \sin \theta\, d\theta}, \quad \int \left(\frac{F(\theta)}{b \sin \theta}\right) \frac{d\theta}{\pi}.$$

Mr. WATSON, in fact, sums the whole number of ways in which the two points can fall on opposite sides of the line, and divides the result by the whole number of ways in which the line and points can fall at all. But I cannot see how this mode of proceeding can be legitimate when the number of possible simultaneous positions of the line and points is different for different inclinations of the line. This number is represented by $b \sin \theta \,.\, \Delta^2$, and as long as it is not constant it cannot be summed independently and used as a divisor to express the probability.

Note on Mr. CARR's *Solution to* Mr. WATSON's *Question* 3693.
By W. S. B. WOOLHOUSE, F.R.A.S.

In compliance with a request from the EDITOR, in regard to Mr. Carr's Solution to Mr. Watson's Question 3693, I am induced to communicate the following brief remarks respecting what is pointed out by Mr. Carr as to the marked discrepancy between his Solution and that previously given by Mr. Watson, the Proposer, and inserted in the *Reprint*, Vol. XVIII., p. 47. Mr. Carr observes that Mr. Watson "sums the whole number of ways in which the two points can fall on opposite sides of the line, and divides the result by the whole number of ways in which the line and points can fall at all." It may at once be stated that not only is this a true account of Mr. Watson's process, but that the method itself is undoubtedly in strict accordance with the fundamental principle of probabilities, and is indeed the most approved and direct way of solving the problem.

The ground of exception urged by Mr. Carr against Mr. Watson's Solution, viz., that the number of possible simultaneous positions of the line and points is different for different inclinations of the line, is indeed the actual source of imperfection in Mr. Carr's method of solution, which, to be correct, would require, as an essential condition, that precisely the same number of combinations should be available for each angle of inclination. To Mr. WATSON's method of procedure no valid objection, as regards accuracy, can possibly arise under any circumstances. Mr. Carr's Solution would be properly adjusted, if, before summing the probabilities appertaining to the several angles, each value were to be multiplied into the facility of arriving at that particular angle; and with such adjustment it would then accurately agree with Mr. Watson's satisfactory Solution of the Question.

Note on Question 3693. By G. S. Carr.

1. Let us consider the following illustrative problem:—There are three boxes A, B, C, containing black and white balls,

A contains 1000 balls, one being black,
B ,, 10 ,, nine ,,
C ,, 10 ,, nine ,,

The probability of drawing a black ball when a box is chosen at random, and a ball is drawn at random from it, is the sum of the separate probabilities of drawing a black ball from each box; viz.,

$$p = \frac{1}{3}\,\frac{1}{1000} + \frac{1}{3}\cdot\frac{9}{10} + \frac{1}{3}\cdot\frac{9}{10},$$

or $$\frac{1}{3}\left\{\frac{1}{1000} + \frac{9}{10} + \frac{9}{10}\right\} = \frac{6}{10} \text{ nearly.}$$

According to the method adopted by Mr. Watson, in his Solution of Question 3693, $$p = \frac{1+9+9}{1000+10+10} = \frac{2}{100} \text{ about,}$$

a result which is clearly wrong.

2. In regard to the Question 3693, "the facility of arriving at any particular angle" (to use the words of Mr. Woolhouse) is $\frac{d\theta}{\pi}$ always; for any inclination of the line is equally probable. In my Solution, I have multiplied the separate probabilities by this factor, for they are not complete without it; and, as this factor is constant, it cannot affect the form of the result.

The above illustrative problem is but one of many such which go to show that the "fundamental principle of probabilities," which Mr. Woolhouse cites, is a principle of very partial application.

The separate probabilities for the different boxes in this problem correspond to the separate probabilities for the different inclinations of the line in Mr. Watson's question; and the factor $\frac{1}{3}$ corresponds to the factor $\frac{d\theta}{\pi}$.

Note on Curvature and Acceleration. *By* Sir James Cockle, F.R.S.

1. Let (ρ, θ) with origin C, and (r, ϑ) with origin O be two systems of polar coordinates to which a plane curve, whereof an element is ds, is referred. Then, equating values of ds^2, we have

$$\rho^2 d\theta^2 + d\rho^2 = r^2 d\vartheta^2 + dr^2 \quad \ldots\ldots\ldots\ldots (1).$$

2. Let $CO = h$, and let ϕ be the angle at which ρ and r meet, then

$$h^2 = \rho^2 + r^2 - 2\rho r \cos\phi \quad \ldots\ldots\ldots\ldots (2).$$

3. Let C be the centre of curvature, then $d\rho = 0$ and $dh = 0$. Make ϑ the independent variable, and (1) gives

$$\rho^2 \frac{d\theta^2}{d\vartheta^2} = r^2 + \frac{dr^2}{d\vartheta^2},\quad \rho^2 \frac{d\theta}{d\vartheta}\,\frac{d^2\theta}{d\vartheta^2} = \frac{dr}{d\vartheta}\left(r + \frac{d^2 r}{d\vartheta^2}\right) \quad \ldots\ldots\ldots (3, 4).$$

4. Now $$\cos\phi = \frac{r}{\rho}\frac{d\vartheta}{d\theta} = \frac{r}{\left(r^2 + \frac{dr^2}{d\vartheta^2}\right)^{\frac{1}{2}}} \quad\text{.....................} (5, 6).$$

Hence, substituting in (2), differentiating and multiplying into $\frac{1}{r}\frac{d\theta^3}{d\vartheta^3}$, we have $$\frac{d\theta}{d\vartheta}\frac{d^2\theta}{d\vartheta^2} = \frac{1}{r}\frac{dr}{d\vartheta}\left(2\frac{d\theta^2}{d\vartheta^2} - \frac{d\theta^3}{d\vartheta^3}\right) \quad\text{........................} (7).$$

5. By means of (7) and (3), we reduce (4) to $$-\rho^2\frac{d\theta^3}{d\vartheta^3} = r\left(r + \frac{d^2r}{d\vartheta^2}\right) - 2\left(r^2 + \frac{dr^2}{d\vartheta^2}\right) \quad\text{..............} (8).$$

But $$-\rho^2\frac{d\theta^3}{d\vartheta^3} = -\frac{1}{\rho}\left(r^2 + \frac{dr^2}{d\vartheta^2}\right)^{\frac{3}{2}} \quad\text{........................} (9),$$

by (3). And (8) gives $$\rho = \frac{\left(r^2 + \frac{dr^2}{d\vartheta^2}\right)^{\frac{3}{2}}}{r^2 + 2\frac{dr^2}{d\vartheta^2} - r\frac{d^2r}{d\vartheta^2}} \quad\text{..........................} (10),$$

the radius of curvature.

6. The acceleration perpendicular to ρ is $\rho\frac{d^2\theta}{dt^2}$ and that along ρ is (velocity)$^2 \div \rho$, or $\rho\frac{d\theta^2}{dt^2}$; for if we treat the element ds as a portion of a circle, then Newton's Prop. IV., Cor. 1 (of the *Principia*) applies.

7. Let R be the acceleration along and P that perpendicular to r. Then $$R\cos\phi - P\sin\phi = \rho\frac{d\theta^2}{dt^2},\quad R\sin\phi + P\cos\phi = \rho\frac{d^2\theta}{dt^2} \quad\text{......} (11, 12).$$

8. Denote by accents differentiations with respect to t. Then $$\rho\theta'^2 = \frac{1}{\rho}(\rho\theta')^2 = \frac{r^2\vartheta'^2 + r'^2}{\rho},\quad \rho\theta'' = \frac{rr'\vartheta'^2 + r^2\vartheta'\vartheta'' + r'r''}{(r^2\vartheta'^2 + r'^2)^{\frac{1}{2}}} \quad\text{......} (13, 14).$$

9. In the ρ of the dexter of (13) change the independent variable from ϑ to t. Then by means of (6) we may give (11) and (12) the respective forms $$Rr\vartheta' - Pr' = rr''\vartheta' - rr'\vartheta'' - r'\vartheta'^3 - 2r'^2\vartheta' \quad\text{...............} (15),$$
$$Rr' + Pr\vartheta' = rr'\vartheta'^2 + r^2\vartheta'\vartheta'' + r'r'' \quad\text{..................} (16);$$

whence $$R = r'' - r\vartheta'^2 = \frac{d^2r}{dt^2} - r\frac{d\vartheta^2}{dt^2} \quad\text{........................} (17),$$

$$P = 2r'\vartheta' + r\vartheta'' = 2\frac{dr}{dt}\frac{d\vartheta}{dt} + r\frac{d^2\vartheta}{dt^2} = \frac{1}{r}\cdot\frac{d}{dt}\left(r^2\frac{d\vartheta}{dt}\right) \quad\text{......} (18).$$

See Thomson and Tait, sect. 32, 36*b*, 330, and 800.

4126. (Proposed by the Rev. T. P. Kirkman, M.A., F.R.S.)—Give an elementary proof of the relation between the numbers which denote the solid angles, faces, and edges of any polyhedron (Euler's theorem).

I. *Solution by the* Rev. Dr. Booth, F.R.S.

Let S, E, F denote the numbers of the solid angles, edges, and faces of any polyhedron; then $S+F=E+2$.

In any *closed* polygon the number of angles is equal to the number of sides, or $s=e$.

In any *open* polygon, or any polygon from which one or more sides have been removed, the number of angles is less by one than the number of sides, or $s_1=e_1-1$.

These are self-evident propositions.

Let us now assume any closed polygon as the base of a proposed polyhedron, and to its sides attach open polygons, and so continue building up as it were the proposed polyhedron by the successive addition of open polygons. We shall at last complete an open polyhedron, or a polyhedron deficient by one face. Let s, s_1, s_2, s_3, s_4, &c. denote the numbers of the angles of the closed polygon, and the angles of the open polygons successively added; in like manner let e, e_1, e_2, e_3, &c. denote the numbers of the sides of the same polygons, then we shall have

$$s=e,\quad s_1=e_1-1,\quad s_2=e_2-1,\quad s_3=e_3-1,\ \ldots\ldots,\ s_n=e_n-1.$$

Now, if we add these numbers, we shall have

$$s+s_1+s_2+s_3\ldots\ldots s_n=S, \text{ and } e+e_1+e_2+e_3\ldots\ldots e_n=E.$$

The sum of the units will manifestly be less by 1 than the number of polygons which constitute the faces of this open polyhedron, or putting F_1 for the number of these faces, we shall have

$$S=E-(F_1-1), \text{ or } S+F_1=E+1.$$

If now we close the aperture of the open polyhedron, by adding one more face, in so doing we shall not alter the numbers of the solid angles or edges; consequently, if F denote the number of the faces of the *closed* polyhedron, we shall have $F_1=F-1$. Substituting this value of F_1 in the preceding expression, we shall have finally $S+F=E+2$. This is Euler's theorem.

It will be obvious that whenever we add on edges to close the serrated contours, as we proceed in building up the polyhedron, for every line we draw connecting any two adjacent vertices of the several polygons, we introduce at the same time an additional face, without altering the number of the solid angles already existing in the incomplete polyhedron; in fact, the line so drawn may be considered as the limiting case of an open polygon, there being one side and no angle.

It will also be evident from the above demonstration that the theorem will still be true when the edges are curve lines, and the faces portions of curved surfaces, or they may be partly plane and partly curved surfaces.

II. *Solution by the* Proposer.

For, suppose that along the edge *ab* of the triangular-faced P one or more triangular sections are possible. Let the section *abc* cut off P′ from P, such that there shall be no edge *de* of P′, along which a triangular section *def* shall cut off from P a less solid than P′. Then no triangular section of P′ along an edge *de* of P is possible, which does not cut the section *abc*; that is, there is an edge *de* of P, along which no triangular section of P is possible.

In such an edge *de*, let *d* run up to *e*. P thus becomes Q, a triangular-faced polyhedron which has three edges, *de*, *df*, *dg*, one summit *d*, and two faces, *def* and *deg*, fewer than P, which has lost as many edges as faces and summits together. Q is a polyhedron; because, as there was no tri-

angular section of P along *de*, there is no linear section of Q through the summit *e*. And Q has every face of P except *def* and *deg*. In like manner, Q becomes R by losing three edges, one summit, and two faces. The final result of such reductions will be a tetrahedron. Thus Euler's theorem about every polyhedron K is proved; for K becomes P by addition of as many diagonals as triangles, and P becomes a tetrahedron, by losing as many edges as faces and summits together.

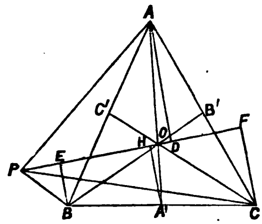
[Our American correspondents inform us that an excellent elementary proof of the relation $E+2 = S+F$ may be found in Chauvenet's *Elementary Geometry*, pp. 235, 236.]

4128. (Proposed by A. B. Evans, M.A.)—Find the locus of a point the sum of the squares of whose distances from the vertices of a given triangle is constant.

I. *Solution by* H. Murphy.

Let the vertices be A, B, C. Bisect AC in D, and let O be a point in the locus; then we have $AO^2+OC^2 = 2AD^2+2DO^2$, therefore $AO^2+OC^2+OB^2 = 2AD^2+2DO^2+OB^2$; but $2AD^2$ is given, hence $2OD^2+BO^2$, and the base DB of the triangle DOB, are given; hence the locus of O is a circle.

II. *Solution by the* Proposer.



Let ABC be the triangle, O the intersection of the medial lines AA′, BB′, CC′, and P one position of the point whose locus is required. Let D, E, F, H be the projections of A, B, C, A′ on PO produced; then, from the triangles APO, BPO, CPO, we have

$$\left.\begin{aligned} AP^2-AO^2 &= PD^2-DO^2 \\ &= PO\,(PO+2DO) \\ BP^2-BO^2 &= PE^2-EO^2 \\ &= PO\,(PE-EO) \\ CP^2-CO^2 &= PF^2-FO^2 \\ &= PO\,(PO+2FO) \end{aligned}\right\} \ldots (1).$$

Since $EH = HF$ and $DO = 2HO$, we obtain, by adding (1),

$$AP^2+BP^2+CP^2-AO^2-BO^2-CO^2 = 3PO^2 \ldots\ldots\ldots\ldots\ldots (2).$$

From (2) it appears that the required locus is a circle whose centre is the centroid of the triangle, and whose radius is

$$\tfrac{1}{3}\left\{AP^2+BP^2+CP^2-AO^2-BO^2-CO^2\right\}^{\frac{1}{2}}$$
$$= \tfrac{1}{3}\left\{3\,(AP^2+BP^2+CP^2)-AB^2-BC^2-AC^2\right\}^{\frac{1}{2}}.$$

[This is a particular case of the general problem, given any number of points to find the locus of a point such that the sum of the squares of the distances, or the multiple squares of the distances, from them to the point may be given.]

4120. (Proposed by Professor Wolstenholme, M.A.)—In a three-cusped hypocycloid whose cusps are A, B, C, if a chord APQ be drawn through A, the tangents at P, Q will divide BC harmonically, and their point of intersection will lie on a conic through BC, the pole of BC being the centre of the hypocycloid.

Solution by Professor Townsend, F.R.S.

This pretty property is true not only for the three-cusped hypocycloid but generally for any tricuspidal quartic, which may be easily shown as follows:—

The equation of the quartic, in trilinear coordinates to the cuspidal as triangle of reference, being

$$\left(\frac{x}{a}\right)^{-\frac{1}{2}} + \left(\frac{y}{b}\right)^{-\frac{1}{2}} + \left(\frac{z}{c}\right)^{-\frac{1}{2}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

[See Salmon's *Higher Plane Curves*, 2nd ed., Art. 284,] those of the two tangents at any two points P and Q, collinear with A, are easily seen to be

$$\left.\begin{aligned} \left(\frac{b}{a}\right)^{\frac{1}{2}} m^{\frac{3}{2}} y + \left(\frac{c}{a}\right)^{\frac{1}{2}} n^{\frac{3}{2}} z &= \left[\left(\frac{b}{a}\right)^{\frac{1}{2}} m^{\frac{1}{2}} + \left(\frac{c}{a}\right)^{\frac{1}{2}} n^{\frac{1}{2}}\right]^3 x \\ \left(\frac{b}{a}\right)^{\frac{1}{2}} m^{\frac{3}{2}} y - \left(\frac{c}{a}\right)^{\frac{1}{2}} n^{\frac{3}{2}} z &= \left[\left(\frac{b}{a}\right)^{\frac{1}{2}} m^{\frac{1}{2}} - \left(\frac{c}{a}\right)^{\frac{1}{2}} n^{\frac{1}{2}}\right]^3 x \end{aligned}\right\} \ldots\ldots\ldots (2),$$

where $m : n =$ the ratio of the perpendiculars from P or Q on AB and AC; putting $x=0$ in the latter, the two lines

$$\left(\frac{b}{a}\right)^{\frac{1}{2}} m^{\frac{3}{2}} y + \left(\frac{c}{a}\right)^{\frac{1}{2}} n^{\frac{3}{2}} z = 0, \quad \left(\frac{b}{a}\right)^{\frac{1}{2}} m^{\frac{3}{2}} y - \left(\frac{c}{a}\right)^{\frac{1}{2}} n^{\frac{3}{2}} z = 0 \ldots\ldots (3),$$

are evidently harmonically conjugate to each other with respect to the two $y=0$, $z=0$, which proves the first part; and eliminating between them the ratio $m : n$, the resulting equation

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} - \frac{z}{c}\right) - 9\left(\frac{x}{a}\right)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

represents evidently a conic touching at B and C the two cuspidal tangents $\left(\frac{x}{a} - \frac{y}{b}\right) = 0$, $\left(\frac{x}{a} - \frac{z}{c}\right) = 0$, which proves the second part.

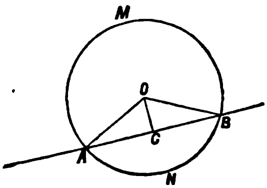
Cor. The three chords of intersection L, M, N, opposite to the three cuspidal tangents, of the three conics U, V, W, connected as above with the three cusps A, B, C, taken in pairs V and W, W and U, U and V, respectively, are easily seen to determine a second triangle A′B′C′ in homology with ABC; the intersection O of the three cuspidal tangents $\left(\frac{y}{b} - \frac{z}{c}\right)$, $\left(\frac{z}{c} - \frac{x}{a}\right)$, $\left(\frac{x}{a} - \frac{y}{b}\right)$, and its polar $\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)$ with respect at once to the quartic and the triangle ABC, being the centre and axis of homology. For, subtracting in pairs the equations of the three conics as given by (4) and its analogues for B and C, we get at once the three equations

$$\left.\begin{aligned} \left[\frac{y}{b} - \frac{z}{c}\right] \cdot \left[4\left(\frac{y}{b} + \frac{z}{c}\right) + \frac{x}{a}\right] &= 0, \\ \left[\frac{z}{c} - \frac{x}{a}\right] \cdot \left[4\left(\frac{z}{c} + \frac{x}{a}\right) + \frac{y}{b}\right] &= 0, \\ \left[\frac{x}{a} - \frac{y}{b}\right] \cdot \left[4\left(\frac{x}{a} + \frac{y}{b}\right) + \frac{z}{c}\right] &= 0, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (5),$$

the first factors of which represent the three cuspidal tangents, and the second the three opposite chords of intersection L, M, N of the three pairs of conics; but, evidently, the former intersect each with the two of the latter to which they do not correspond, and the latter intersect with the corresponding sides of the triangle ABC at their intersections with the line $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$; and therefore, &c.

4066. (Proposed by Professor Crofton, F.R.S.)—A straight line drawn at random crosses a circle; if two points are taken at random within the circle, show that the chance that they fall on opposite sides of the line is $\frac{128}{45\pi^2}$.

Solution by C. Leudesdorf.



The chance that the line makes an angle ϕ with a fixed direction, and that it lies at a distance p from the centre of the circle is

$\frac{d\phi}{\pi}\frac{dp}{2a}$ (if a be the radius of the circle)

$$= -\frac{d\phi}{\pi}\sin\theta\,\frac{d\theta}{2},$$

if 2θ be the angle AOB subtended by AB at the centre O.

And the chance that the two points taken lie on opposite sides of AB, is

$$\frac{2\text{AMB}\,.\,\text{ANB}}{(\pi a^2)^2}=\frac{2}{\pi^2 a^4}\left\{\tfrac{1}{2}a^2(2\theta-\sin 2\theta)\left[\pi a^2-\tfrac{1}{2}a^2(2\theta-\sin 2\theta)\right]\right\};$$

therefore the chance required is

$$=\frac{1}{4\pi^2}\int_0^\pi\frac{d\phi}{\pi}\int_0^\pi\sin\theta\,(2\theta-\sin 2\theta)\,(2\pi-2\theta+\sin 2\theta)\,d\theta$$

$$=\frac{1}{4\pi^2}\int_0^\pi\sin\theta\,(2\theta-\sin 2\theta)\,(2\pi-2\theta+\sin 2\theta)\,d\theta$$

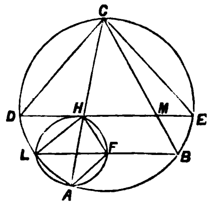
$$=\frac{2}{\pi^2}\int_0^\pi\sin^2\theta\cos\theta\,(\pi-2\theta+\sin 2\theta)\,d\theta,\ \text{(integrating by parts,)}$$

$$=\frac{8}{3\pi^2}\int_0^\pi\sin^5\theta\,d\theta,\ \text{(integrating by parts again,)}$$

$$=\frac{8}{3\pi^2}\times\frac{16}{15}=\frac{128}{45\pi^2}.$$

3867. (Proposed by A. B. EVANS, M.A.)—From two given points in the circumference of a given circle to draw two straight lines to a point in the same circumference, which shall cut a straight line given in position, so that the part of it intercepted by the two lines may be (1) of a given length, or (2) a maximum.

Solution by H. MURPHY; *the* PROPOSER; *and others.*



1. Let A, B be the given points; DE the line given in position. Draw BL parallel to ED, and make BF equal the line to be intercepted. Upon AF draw a segment AHF of a circle similar to ACB; produce AH to C; then C will be the point required. For FM is a parallelogram, and therefore HM is equal to FB.

2. The intercept HM will have its greatest length when the segment on AF touches DE, and DH will then be equal to ME. For as the angle AFL or its equal AHL increases, the line BF or its equal HM also increases; but AHL is a maximum when the segment on AF touches DE; therefore HM is then also a maximum. Again, because the points D, A, B, E are fixed, the anharmonic ratio of the points D, H, M, E is also given; that is, DE . HM : DH . ME is given; but DE . HM is a maximum; therefore DH . ME is also a maximum; and since their sum is a minimum (the maximum line HM being taken from the given line DE), therefore DH = ME, by elementary principles.

4069. (Proposed by Professor CLIFFORD.)—1. Curves of order $2n+1$ pass n times through each circular point, and through n^2+4n+1 other fixed single points (or their equivalent in multiple points); show that the envelope of their asymptotes is a tricusped hypocycloid.

2. Curves of order $2n+2$ pass n times through each circular point and through n^2+6n+4 other fixed points, and their real asymptotes are at right angles; show that the envelope of their asymptotes is a tricusped hypocycloid.

Solution by Professor WOLSTENHOLME, M.A.

1. This is not quite true as it stands, as it is necessary that the factor x^2+y^2 shall enter in the nth degree in the highest terms, and also in the $(n-1)$th degree in the next terms.

The number of given conditions is one less than sufficient to determine a curve of this order, hence if U, V be any two of the curve, the general form is $U+\lambda V=0$, and if we take the real asymptotes of U, V as axes of x and y, the form of the general equation will be

$$U+\lambda V=0,$$

where $U \equiv x(x^2+y^2)^n + x(x^2+y^2)^{n-1}(ax+by) +$ terms of lower dimensions,

$$V \equiv y(x^2+y^2)^n + y(x^2+y^2)^{n-1}(a'x+b'y) + \quad \text{,,} \qquad \text{,,}$$

and the equation of the real asymptote of $U+\lambda V=0$ is $x+\lambda y=p$, where

$$p(1+\lambda^2)^n+(1+\lambda^2)^{n-1}\left\{\lambda^2(a-a')-\lambda(b-b')\right\} = 0,$$

or the equation of the asymptote is

$$(x+\lambda y)(1+\lambda^2)+\lambda^2(a-a')-\lambda(b-b') = 0,$$

or if $\lambda \equiv \tan\theta$,

$$x\cos\theta+y\sin\theta+(a-a')\sin^2\theta\cos\theta-(b-b')\cos^2\theta\sin\theta = 0,$$

or $\quad 4(x\cos\theta+y\sin\theta)+(a-a')(\cos\theta-\cos 3\theta)-(b-b')(\sin 3\theta+\sin\theta)$,

which, by a change of origin and axes, may be written

$$x\cos\theta+y\sin\theta = \cos 3\theta,$$

whose envelope is a tricusped hypocycloid.

2. Treating this similarly, we may take

$$U \equiv (x^2+y^2)^n(x^2+kxy-y^2)+U_{2n},$$
$$V \equiv (x^2+y^2)^n\left\{(x-a)^2+k'(x-a)(y-b)-(y-b)^2\right\}+V_{2n},$$

and the real asymptotes are the asymptotes of the rectangular hyperbola

$$x^2+kxy-y^2+\lambda\left\{(x-a)^2+k'(x-a)(y-b)-(y-b)^2\right\} = 0,$$

which is a rectangular hyperbola through four given points (each being the centre of perpendiculars of the triangle formed by the other three), and the envelope of these asymptotes is a tricusped hypocycloid, as has been already proved in the *Educational Times.*

In (2), the n^2+6n+4 given points are not all arbitrary, any one being determined when the other n^2+6n+3 are known.

4107. (Proposed by John C. Malet, B.A.)—Solve the biquadratic equation $x^4+px^2+qx+r=0$ by direct substitution for x of a function of another unknown y.

Solution by the Proposer.

If, in the biquadratic equation

$$x^4+px^2+qx+r = 0,$$

we make the substitution

$$x = -y \pm \left(\frac{q}{4y}-\frac{p}{2}-y^2\right)^{\frac{1}{2}},$$

we find, after a little reduction, Euler's reducing cubic in y^2,

$$y^6+\tfrac{1}{2}py^4+\tfrac{1}{16}(p^2-4r)y^2-\tfrac{1}{64}q^2 = 0.$$

Hence, if we find a root of this cubic, we can at once write down two roots of the biquadratic.

In some cases this method is shorter than the usual methods of solving

a biquadratic equation which wants its second term; for example, take the equation $x^4+px^2+qx+\frac{1}{4}\left\{1+p+q\right\}\left\{1+p-q\right\}=0.$

It is easily seen that one value of y is $\frac{1}{2}$, and therefore we have at once two roots of the biquadratic, namely,

$$\tfrac{1}{2}\left\{-1\pm[2(q-p)-1]^{\frac{1}{2}}\right\}.$$

4093. (Proposed by Professor CROFTON, F.R.S.)—1. Show that the mean square of the distance of a fixed point within any closed convex contour, from the perimeter, is constant wherever the point be taken, and equal to the square of the radius of a circle of the same area as the contour,—the distances being taken at equal angular intervals.

2. Show that the mean square of the distance between two points taken at random in any convex area, is twice the square of the radius of gyration of the area round its centre of gravity.

Solution by Professor WOLSTENHOLME, M.A.

1. The mean square for equal angular distances $=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} r^2\,d\theta$, and $\displaystyle\int_0^{2\pi} r^2 d\theta =$ twice the area of the curve whatever be the origin, hence the mean square $=\dfrac{1}{\pi}\times$ Area $=$ square on the radius of a circle of the same area.

2. This origin being at the centre of gravity (x, y), (x', y') any two points taken at random within the contour, the mean square of the distance between them is

$$\iiiint\left(\overline{x-x'}^2+\overline{y-y'}^2\right)dx\,dy\,dx'\,dy' \text{ divided by } \iiiint dx\,dy\,dx'\,dy',$$

and this denominator is the square of the area. Also the numerator is, since the origin is the centre of inertia,

$$\iiiint(x^2+y^2+x'^2+y'^2)\,dx\,dy\,dx'\,dy'$$

$$=\text{Area}\times\left(\iint(x^2+y^2)\,dx\,dy+\iint(x'^2+y'^2)\,dx'\,dy'\right)$$

$=2$ Area $\times$ moment of inertia,

$=2(\text{Area})^2\times$ square on radius of gyration about an axis perpendicular to the plane.

4120. (Proposed by Professor WOLSTENHOLME, M.A.)—In a three-cusped hypocycloid, whose cusps are A, B, C, if a chord APQ be drawn

through A, the tangents at P, Q will divide BC harmonically, and their point of intersection will lie on a conic through BC, the pole of BC being the centre of the hypocycloid.

4142. (Proposed by Professor WOLSTENHOLME, M.A.)—If a tangent to a cardioid meet the curve again in PQ, the tangents at P, Q divide the double tangent harmonically, and the locus of their point of intersection is a conic through the points of contact of the double tangent and having triple contact with the cardioid (two of the contacts impossible).

Solution by Professor TOWNSEND, F.R.S.

These two pretty companion properties, which, as projective, are true for all tricuspidal quartics as well as for the above [which may be regarded as the simplest representatives of the two different classes into which they may be divided according as the three cusps are all real or two of them imaginary], may be readily inferred by projection, that for each representative curve from a simple well known property of the other, as follows:—

In the cardioid, whose two imaginary cusps are the two circular points at infinity, the tangents at the extremities of any chord passing through the real cusp intersect, as is well known, at right angles on the circle concentric with the focal circle which passes through the vertex of the curve, and being rectangular divide harmonically the segment intercepted between the two imaginary cusps; hence, by projection, in any tricuspidal quartic, and therefore in the three-cusped hypocycloid, the tangents at the extremities of any chord passing through a real cusp divide harmonically the segment, real or imaginary, intercepted between the two remaining cusps, and intersect on the conic which touches the curve at the two latter cusps and at its intersection with the tangent at the original cusp; which is the first property. And, in the three-cusped hypocycloid, whose double tangent is at infinity and touches at the two circular points, the tangents at the extremities of any chord touching the curve intersect, as is also well known, at right angles on the circle inscribed to the curve, and being rectangular divide harmonically the segment intercepted between the two imaginary points of contact of the double tangent; hence, by projection, in any tricuspidal quartic, and therefore in the cardioid, the tangents at the extremities of any chord touching the curve divide harmonically the segment, real or imaginary, intercepted between the two points of contact of the double tangent, and intersect on the conic which passes through those two points of contact and touches the curve at its three intersections with the three cuspidal tangents; which is the second property.

4116. (Proposed by the Rev. J. L. KITCHIN, M.A.)—Prove that

$$1 + \frac{2^3}{\lfloor 2} + \frac{3^3}{\lfloor 3} + \frac{4^3}{\lfloor 4} + \ldots\ldots = 5e, \qquad 1 + \frac{2^4}{\lfloor 2} + \frac{3^4}{\lfloor 3} + \frac{4^4}{\lfloor 4} + \ldots\ldots = 15e,$$

and find the value of $\quad 1 + \dfrac{2^n}{\lfloor 2} + \dfrac{3^n}{\lfloor 3} + \dfrac{4^n}{\lfloor 4} + \ldots\ldots$

I. *Solution by* S. FORDE.

Because $$f(x) = f(0) + \Delta f(0) \, . \, x + \frac{\Delta^2 f(0)}{\underline{|2}} x^2 + \dots,$$

therefore $$x^n = \Delta 0^n . + \frac{\Delta^2 0^n}{\underline{|2}} x^2 + \frac{\Delta^3 0^n}{\underline{|3}} x^3 + \dots;$$

therefore $$\frac{x^n}{\underline{|x}} = \frac{\Delta 0^n}{\underline{|x-1}} x + \frac{\Delta^2 0^n}{\underline{|x-2}\,\underline{|2}} x^2 + \frac{\Delta^3 0^n}{\underline{|x-3}\,\underline{|3}} x^3 + \dots;$$

therefore $$\Sigma_1^\infty \frac{x^n}{\underline{|x}} = \left(\Delta 0^n + \frac{\Delta^2 0^n}{\underline{|2}} + \frac{\Delta^3 0^n}{\underline{|3}} + \dots\right) e = e\,[(e^\Delta - 1)\, 0^n].$$

If $n=3$, $$\Delta 0^n = 1, \quad \Delta^2 0^n = 6, \quad \Delta^3 0^n = 6,$$

therefore $$\Sigma \frac{x^3}{\underline{|x}} = e\left(1 + \frac{6}{\underline{|2}} + \frac{6}{\underline{|3}}\right) = 5e.$$

If $n=4$, $$\Delta 0^n = 1, \; \Delta^2 0^n = 14, \; \Delta^3 0^n = 36, \; \Delta^4 0^n = 24,$$

therefore $$\Sigma \frac{x^4}{\underline{|4}} = e\left(1 + \frac{14}{\underline{|2}} + \frac{36}{\underline{|3}} + \frac{24}{\underline{|4}}\right) = 15e.$$

II. *Solution by* Professor WOLSTENHOLME, M.A.

$$y \equiv \epsilon^{\epsilon^x} = 1 + \epsilon^x + \frac{\epsilon^{2x}}{\underline{|2}} + \frac{\epsilon^{3x}}{\underline{|3}} + \dots\dots,$$

hence if $u_n \equiv 1^n + \frac{2^n}{\underline{|2}} + \frac{3^n}{\underline{|3}} + \dots\dots$, u_n is $\underline{|n}$ times the coefficient of x^n in the expansion of ϵ^{ϵ^x}, and is therefore the value of $\frac{d^n y}{dx^n}$ when $x=0$. Many series may readily be found for determining this; thus we have

$$\frac{dy}{dx} = y\epsilon^x; \text{ therefore } \frac{d^{n+1}y}{dx^{n+1}} = \epsilon^x \left(y + n\frac{dy}{dx} + \frac{n(n-1)}{2}\frac{d^2y}{dx^2} + \dots + \frac{d^n y}{dx^n}\right),$$

hence $$u_{n+1} = \epsilon + nu_1 + \frac{n(n-1)}{\underline{|2}} u_2 + \dots + u_n \quad \dots\dots\dots\dots\dots\dots (1).$$

Again, $y = \epsilon^{-x}\frac{dy}{dx}$; therefore

$$\frac{d^n y}{dx^n} = \epsilon^{-x}\left(\frac{d^{n+1}y}{dx^{n+1}} - n\frac{d^n y}{dx^n} + \frac{n(n-1)}{2}\frac{d^{n-1}y}{dx^{n-1}} - \dots + (-1)^n \frac{dy}{dx}\right);$$

whence $$u_n = u_{n+1} - nu_n + \frac{n(n-1)}{\underline{|2}} u_{n-1} - \dots + (-1)^n u_1 = \Delta^n u_1 \quad \dots (2).$$

Now $$u_1 = 1 + 1 + \frac{1}{\underline{|2}} + \dots = \epsilon;$$

therefore by (1) $u_2 = \epsilon + u_1 = 2\epsilon$, therefore $u_3 = \epsilon + 2u_1 + u_2 = 5\epsilon$,

$$u_4 = \epsilon + 3u_1 + 3u_2 + u_3 = \epsilon(1 + 3 + 6 + 5) = 15\epsilon.$$

It is obvious that u is always a multiple of ϵ. There is no difficulty in

proving these series without the calculus; for writing

$$u_n = \Sigma_{r=1}^{r=\infty} \frac{r^{n-1}}{\lfloor r-1},$$

we have $$u_{n+1} = \Sigma \frac{r^n}{\lfloor r-1}.$$

$$= \Sigma \left\{ \frac{(r-1)^{n-1}}{\lfloor r-2} + n \cdot \frac{(r-1)^{n-2}}{\lfloor r-2} + \frac{n(n-1)}{\lfloor 2} \frac{(r-1)^{n-3}}{\lfloor r-2} + \ldots \frac{n}{\lfloor r-2} + \frac{1}{\lfloor r-1} \right\}$$

$$= u_n + nu_{n-1} + \frac{n(n-1)}{\lfloor 2} u_{n-2} + \ldots + nu_1 + \epsilon,$$

which is (1), and similarly by writing $r+1-1$ for r and expanding we may obtain (2).

4005. (Proposed by T. P. Kirkman, M.A., F.R.S.)—

Z, being rich, in his mansion fine,
The rest of the letters invites to dine,
For every day
Of merry May,
Whenever they choose; but begs to fix
Conditions two—that never twain
Who meet at his board shall meet again,
And that never the guests be more than six.

They go in sixes, they go in fives,
Feeding as never they fed in their lives;
Then four and four,
Till the month was o'er,
And so that none
Got dinners more
Than other one.
Now show what your art is,
By naming the parties.

Solution by the Proposer.

The arrangement of the parties will be as follows:—

ABKL	FEWY	CEQS
AFOP	FHTY	GJNO
AIRS	IDWX	GHQR
BCUV	IJTU	EJML
BGXY	CDNP	HDMK,

BFIQMN	BJHSWP
ACGWMT	FJCXRK
AEHUNX	ICHLYO
AJDYQY	FGDLUS
BDERTO	IGEKVP,

MPRUY	KOQUW	LPQTX
MOSVX	KNSTY	LNRVW.

4095. (Proposed by Professor Wolstenholme, M.A.)—An ellipse and an hyperbola are confocal, and the asymptotes of the one are the equal

conjugate diameters of the other; from any point O on the one tangents OP, OQ are drawn to the other, and the tangent at O meets the second curve in R, R′; prove that R, R′ are centres of two of the four circles which touch the sides of the triangle OPQ. [The Proposer remarks that the curves may also be confocal, equal, and opposite parabolas.]

Solution by the PROPOSER.

The equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is equivalent to

$$\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha-1\right)\left(\frac{x}{a}\cos\beta+\frac{y}{b}\sin\beta-1\right)$$
$$=\left(\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}\right)^2,$$

whence if p_1, p_2, p_3 be the perpendiculars from any point of the ellipse on the tangents at points whose excentric angles are α, β, and on the chord of contact,
$$p_1^2p_2^2.\left(\frac{\cos^2\alpha}{a^2}+\frac{\sin^2\alpha}{b^2}\right)\left(\frac{\cos^2\beta}{a^2}+\frac{\sin^2\beta}{b^2}\right)$$
$$=p_3^4\left(\frac{1}{a^2}\cos^2\frac{\alpha+\beta}{2}+\frac{1}{b^2}\sin^2\frac{\alpha+\beta}{2}\right)^2.$$

Hence $p_1p_2=p_3^2$ if $\left(\dfrac{1}{a^2}\cos^2\alpha+\dfrac{1}{b^2}\sin^2\alpha\right)\left(\dfrac{1}{a^2}\cos^2\beta+\dfrac{1}{b^2}\sin^2\beta\right)$
$$=\left(\frac{1}{a^2}\cos^2\frac{\alpha+\beta}{2}+\frac{1}{b^2}\sin^2\frac{\alpha+\beta}{2}\right)^2,$$

which equation is equivalent to

$$\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\sin^2\frac{\alpha-\beta}{2}\left\{\frac{1}{a^2}\left(\cos^2\frac{\alpha-\beta}{2}+\cos(\alpha+\beta)\right)\right.$$
$$\left.-\frac{1}{b^2}\left(\cos^2\frac{\alpha-\beta}{2}-\cos(\alpha+\beta)\right)\right\}=0,$$

of which the last factor gives us the locus of the point of intersection of the tangents

$$\frac{1}{a^2}\left(1+\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-\frac{1}{b^2}\left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right),\ \text{or}\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{a^2-b^2}{a^2+b^2},$$

an hyperbola confocal with the ellipse and having its asymptotes along the conjugate diameters.

Hence, if tangents be drawn from any point on this hyperbola, since for any point on the ellipse $p_1p_2=p_3^2$, the ellipse will pass through the centres of the inscribed circle and the escribed circle opposite the chord of contact, and these centres will lie on the straight line bisecting the angle between the tangents to the ellipse; which, since the hyperbola is confocal with the ellipse, will be the tangent to the hyperbola. I once investigated the locus of the two other centres, but rather think it turned out to be of the eighth degree. If we move the origin to a focus, and make a, b infinite, and $\dfrac{b^2}{a}=2m$, the two equations become in the limit $y^2=4m(x+m)$ and $y^2=4m(m-x)$, two equal and opposite confocal parabolas. This may of course be proved directly in the same way as above, the equation of the parabola $y^2=4ax$, being equivalent to

$$4(my-m^2x-a)(m'y-m'^2x-a) = \left\{y(m+m')-2mm'x-2a\right\}^2,$$

or $$16m^2m'^2(1+m^2)(1+m'^2)p_1^2p_2^2 = \left\{(m+m')^2+4m^2m'^2\right\}^2 p_3^4;$$

so that $p_1p_2 = p_3^2$ if $16m^2m'^2(1+m^2)(1+m'^2) = \left\{(m+m')^2+4m^2m'^2\right\}^2$,

which is equivalent to $(m-m')^2\left\{(m+m')^2+4mm'-8m^2m'^2\right\}^2 = 0$;

and since the point of intersection of the two tangents is

$$X = \frac{a}{mm'},\quad Y = a\left(\frac{1}{m}+\frac{1}{m'}\right),$$

the locus of their intersection will be

$$Y^2+4aX-8a^2 = 0,$$

a parabola equal to and confocal with the original, and, as before, the two centres will be the points where the tangent to the second parabola meets the first.

I have found the form of the equation of the ellipse used above exceedingly useful in the investigation of properties of two tangents and their chord of contact; as also the form

$$\left(\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}\right)\left(\frac{x}{a}\cos\frac{\gamma+\delta}{2}+\frac{y}{b}\sin\frac{\gamma+\delta}{2}-\cos\frac{\gamma-\delta}{2}\right)$$
$$=\left(\frac{x}{a}\cos\frac{\alpha+\gamma}{2}+\frac{y}{b}\sin\frac{\alpha+\gamma}{2}-\cos\frac{\alpha-\gamma}{2}\right)\left(\frac{x}{a}\cos\frac{\beta+\delta}{2}+\frac{y}{b}\sin\frac{\beta+\delta}{2}-\cos\frac{\beta-\delta}{2}\right)$$
$$=\left(\frac{x}{a}\cos\frac{\alpha+\delta}{2}+\frac{y}{b}\sin\frac{\alpha+\delta}{2}-\cos\frac{\alpha-\delta}{2}\right)\left(\frac{x}{a}\cos\frac{\beta+\gamma}{2}+\frac{y}{b}\sin\frac{\beta+\gamma}{2}-\cos\frac{\beta-\gamma}{2}\right);$$

which shows that, if from any point of the ellipse $p_1, p_1', p_2, p_2', p_3, p_3'$ be the perpendiculars let fall on the three pairs of chords joining a quadrangle, and $r_1, r_1', r_2, r_2', r_3, r_3'$ the parallel central radii,

$$p_1p_1'r_1r_1' = p_2p_2'r_2r_2' = p_3p_3'r_3r_3';$$

or if ϖ_1, ϖ_1', &c. be the central perpendiculars on the parallel tangents,

$$\frac{p_1p_1'}{\varpi_1\varpi_1'} = \frac{p_2p_2'}{\varpi_2\varpi_2'} = \frac{p_3p_3'}{\varpi_3\varpi_3'};$$

a result which is proved in SALMON'S *Conic Sections*, and which is immediately obvious from orthogonal projection of a circle.

3995. (Proposed by W. H. H. HUDSON, M.A.)—A fixed horizontal rod OQ carries a small smooth ring of mass M′ upon it. A string is fastened at O, passes through the ring, and is attached to a particle of mass M. Initially the particle is at rest just outside the ring, and the string is straight. Obtain equations to determine the motion; and find the initial tension of the string and the initial acceleration of the ring.

Solution by G. S. CARR.

Let A be the initial position of the ring and particle. Take A for origin, AO for the axis of x and the axis of y downwards. Let x be the

distance of the ring from A, and hk the coordinates of the particle at the time t. Also let the angle MM′A $= \theta$, and tension of string $=$ T.

The dynamical equations will be

$$M'\frac{d^2x}{dt^2} = T(1-\cos\theta), \quad M\frac{d^2h}{dt^2} = T\cos\theta, \quad M\frac{d^2k}{dt^2} = Mg - T\sin\theta \ldots (1).$$

The geometrical equations are

$$h = x \operatorname{vers}\theta, \quad k = x\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Multiplying equations (1) by $2\frac{dx}{dt}, 2\frac{dh}{dt}, 2\frac{dk}{dt}$ respectively, adding and integrating the result, from which T disappears by equations (2), we obtain

$$M'\left(\frac{dx}{dt}\right)^2 + M\left(\frac{dh}{dt}\right)^2 + M\left(\frac{dk}{dt}\right)^2 = 2mkg,$$

the constant vanishing with t.

Substitute for $\frac{dh}{dt}, \frac{dk}{dt}$ from (2), and we have

$$M'(dx)^2 + M\left\{2\operatorname{vers}\theta\,(dx)^2 + 2x\sin\theta\,dx\,d\theta + x^2(d\theta)^2 - 2gx\sin\theta\,(dt)^2\right\} = 0,$$

a differential equation for the path described by the particle M.

For the initial motion, since the particle must hang vertically from the ring at the outset, we have $\theta = \frac{1}{2}\pi$ and $\frac{d^2k}{dt^2} = \frac{d^2x}{dt^2}$. Therefore by equations (1)

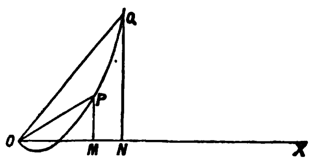
$$M'\frac{d^2x}{dt^2} = T, \quad M\frac{d^2x}{dt^2} = Mg - T;$$

from which we find $\frac{d^2x}{dt^2} = \frac{Mg}{M+M'}$ and $T = \frac{MM'g}{M+M'}$,

the initial acceleration and tension required.

4059. (Proposed by J. L. Mackenzie.)—Let OX be a fixed line, POX a variable angle, POQ an angle exceeding POX by a constant given difference; PM and QN perpendiculars to OX. If the lengths of OP and OQ are so determined that ON has a constant given ratio to OM, find the curve which passes through all such points as P and Q.

Solution by the Proposer.



Let $POM = \theta = u_x$,

$QON = 2\theta + a = u_{x+1}$,

$OM = r\cos\theta = v_x$,

$ON = r\cos(2\theta + a) = v_{x+1}$.

Then, by the conditions of the problem, $v_{x+1} = mv_x$.

The solution of this difference-equation is

$$v_x = c_1 m^x, \text{ or } x \log m = \log \frac{v_x}{c_1} \text{ .... (1).}$$

Again, $$u_{x+1} - 2u_x = a,$$

whence $$u_x + a = c_2 2^x, \text{ or } x \log 2 = \log \frac{u_x + a}{c_2} \text{ .... (2).}$$

Eliminating x between (1) and (2), and putting $n = \dfrac{\log m}{\log 2}$, we have

$$v_x = \frac{c_1}{c_2^n}(u_x + a)^n, \text{ or } r \cos\theta = k(\theta + a)^n;$$

where k is any function of θ, which does not change, if θ is changed into $2\theta + a$. The most general expression for k is found to be

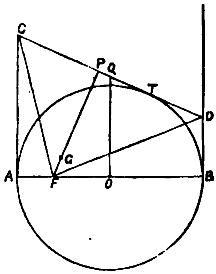
$$A_0 + A_1 \cos\left\{\frac{2\pi \log(\theta + a)}{\log 2}\right\} + A_2 \cos\left\{\frac{4\pi \log(\theta + a)}{\log 2}\right\} + \text{\&c.},$$

$$+ B_0 + B_1 \sin\left\{\frac{2\pi \log(\theta + a)}{\log 2}\right\} + B_2 \sin\left\{\frac{4\pi \log(\theta + a)}{\log 2}\right\} + \text{\&c.},$$

where A_0, A_1, A_2 ...... B_0, B_1, B_2, &c., are arbitrary constants.

4166. (Proposed by the Editor.)—If AB be the major axis of an ellipse, and PFQ a focal chord, prove that FP tan APB = FQ tan AQB.

Solution by the Rev. Dr. Booth, F.R.S.



This theorem is the *reciprocal polar* of the following property of the circle.

Let a point F be assumed in the diameter AB of a circle. Let tangents to it be drawn through A and B, and the transverse tangent CD, touching the circle in T. On this tangent let fall the perpendicular FP, and erect through O, the centre of the circle, OQ perpendicular to AB. Then the circle, whose centre is Q and radius QO will pass through the points C, D, G and will touch AB in O.

Then $\tan CFD = \dfrac{FP \cdot CD}{(FP)^2 - CP \cdot DP}$, but $CP \cdot DP = (PG)^2$, and $(FP)^2 - (PG)^2 = (FO)^2$, a constant. Let R be the radius of the polarising circle, and let f be the corresponding focal chord drawn through F the focus of the ellipse; then $FP = \dfrac{R^2}{f}$, consequently $f \tan CFD = \dfrac{R^2 CD}{(FO)^2}$. We should find the same value for $f' \tan C'FD'$; therefore

$$f \tan CFD = f' \tan C'FD'.$$

Now the reciprocal polar of the circle ATB, the centre of the polarising circle being at F, is an ellipse whose focus is at F, the poles of the tangents AC, CD, BD being the extremities of the major axis of the reciprocal ellipse, and the point to which the chords in the ellipse are drawn; while C and D are the poles of these chords, and CFD is equal to the angle between these chords.

The algebraical proof of this theorem is very simple. Assuming the usual coordinates through the centre of the curve, let ϕ be the angle between the chords, and y the ordinate of the point on which they meet on the curve. Then $\tan\phi = \dfrac{2b^2}{ae^2y}$, or $y\tan\phi = \dfrac{2b^2}{ae^2}$.

In like manner $y_1\tan\phi_1 = \dfrac{2b^2}{ae^2}$, but $y : y_1 = f : f_1$ and p the parameter is equal to $\dfrac{2b^2}{a}$; consequently, $f\tan\phi = \dfrac{p}{e^2}$, and $f_1\tan\phi_1 = \dfrac{p}{e^2}$.

II. *Solution by* R. Tucker, M.A.

Let P′, Q′ be the corresponding points to P, Q on the auxiliary circle; then if AP′, AQ′ make angles θ, θ' respectively with AB, we can readily show that
$$\tan \mathrm{APB} = -\frac{b}{ae^2}\cdot\frac{1}{\sin\theta\cos\theta};$$
therefore
$$\frac{\tan \mathrm{APB}}{\tan \mathrm{AQB}} = \frac{\sin\theta'\cos\theta'}{\sin\theta\cos\theta} = \frac{\mathrm{AQ'}.\mathrm{BQ'}}{\mathrm{AP'}.\mathrm{BP'}} = \frac{\mathrm{Q'N'}}{\mathrm{P'N}}$$
(if N, N′ are the projections of P and Q on AB) $= \dfrac{\mathrm{QN}}{\mathrm{PN}} = \dfrac{\mathrm{FQ}}{\mathrm{FP}}$;

therefore
$$\mathrm{FP}\tan \mathrm{APB} = \mathrm{FQ}\tan \mathrm{AQB}.$$

4008. (Proposed by Professor Wolstenholme.)—Solve the equations
$10y^2+13z^2-6yz = 242,\quad 5z^2+10x^2-2zx = 98,\quad 13x^2+5y^2-16xy = 2,$
proving that an infinite number of solutions exists.

Solution by S. Bills; H. Murphy; *and others.*

From (1), $y^2-\frac{3}{5}zy = \frac{121}{5}-\frac{13}{10}z^2$, whence $y = \frac{1}{10}\left\{32\pm 11\,(20-z^2)^{\frac{1}{2}}\right\}$;
also from (2), $x^2-\frac{1}{5}zx = \frac{49}{5}-\frac{1}{2}z^2$, whence $x = \frac{1}{10}\left\{2\pm 7\,(20-z^2)^{\frac{1}{2}}\right\}$.

Substituting these values of y and x in (3), it will be found to vanish identically, which shows that the equations admit of an infinite number of solutions.

[The equation $11x-7y+z = 0$ combined with any one of the three will lead to the other two, proving that the three cylinders represented by these equations have a common plane section, and therefore an infinite number of points common. Any two will therefore intersect in two planes; for instance, taking the first and second equations, we have either $11x-7y+z = 0$, or $55x+35y-16z = 0$; and we shall thus get, besides the points lying on the common section, two other points, that is, solutions of the three equations.]

4026. (Proposed by A. MARTIN.)—There is an integer series of right-angled triangles whose legs differ by unity only, that whose sides are 3, 4, 5 being the first triangle; find expressions for the sides of the nth triangle, and show that the sides of the 80th triangle are

10588278309438211127768625972711138460195892610538807320361440,
10588278309438211127768625972711138460195892610538807320361441,
14974086787388384990495417211933241811765094618559069827415009.

I. *Solution by the* PROPOSER.

Let $\frac{1}{2}(x-1)$, $\frac{1}{2}(x+1)$, and y denote the sides of such a triangle. Then we have
$$\tfrac{1}{4}(x-1)^2+\tfrac{1}{4}(x+1)^2 = y^2;$$
whence
$$x^2-2y^2 = (x-y\sqrt{2})(x+y\sqrt{2}) = -1 \quad\text{.................. (1, 2).}$$
Let p and q be known values of x and y; then $p^2-2q^2=-1$, and
$$(p-q\sqrt{2})^{2n+1}\,(p+q\sqrt{2})^{2n+1} = -1 \quad\text{.................. (3).}$$
Assuming $x_n - y_n\sqrt{2} = (p-q\sqrt{2})^{2n+1}$, $x_n+y_n\sqrt{2} = (p+q\sqrt{2})^{2n+1}$,
we obtain
$$x_n = \tfrac{1}{2}\left[(p+q\sqrt{2})^{2n+1}+(p-q\sqrt{2})^{2n+1}\right],$$
$$y_n = \frac{1}{2\sqrt{2}}\left[(p+q\sqrt{2})^{2n+1}-(p-q\sqrt{2})^{2n+1}\right],$$
where n may be 0, 1, 2, 3, 4, &c.

It is easily seen that (1) is satisfied by $x=1$, $y=1$; take, therefore, $p=1$, $q=1$, and we have
$$x_n = \tfrac{1}{2}\left[(1+\sqrt{2})^{2n+1}+(1-\sqrt{2})^{2n+1}\right],$$
$$y_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{2n+1}-(1-\sqrt{2})^{2n+1}\right];$$
and the sides of the nth triangle are
$$\left.\begin{aligned}\tfrac{1}{2}(x_n-1) &= \tfrac{1}{4}\left[(1+\sqrt{2})^{2n+1}+(1-\sqrt{2})^{2n+1}-2\right],\\ \tfrac{1}{2}(x_n+1) &= \tfrac{1}{4}\left[(1+\sqrt{2})^{2n+1}+(1-\sqrt{2})^{2n+1}+2\right],\\ y_n &= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{2n+1}-(1-\sqrt{2})^{2n+1}\right].\end{aligned}\right\} \quad\text{......... (4).}$$

The operation of involution is very laborious when n is a large number.

The numerical values of x and y that satisfy (1) are the numerators and denominators of the odd convergents to the square root of 2 expanded as a continued fraction. These odd convergents are
$$\frac{1}{1},\ \frac{7}{5},\ \frac{41}{29},\ \frac{239}{169},\ \frac{1393}{985},\ \text{\&c.}$$
Writing $\dfrac{x}{y}$, $\dfrac{x_1}{y_1}$, $\dfrac{x_2}{y_2}$, $\dfrac{x_3}{y_3}$, $\dfrac{x_4}{y_4}$, &c., for the above fractions, we have
$$\left.\begin{aligned}x_2 &= 6x_1-x\\ y_2 &= 6y_1-y\end{aligned}\right\},\quad \left.\begin{aligned}x_3 &= 6x_2-x_1\\ y_3 &= 6y_2-y_1\end{aligned}\right\},\ \ldots\ldots\quad \left.\begin{aligned}x_n &= 6x_{n-1}-x_{n-2}\\ y_n &= 6y_{n-1}-y_{n-1}\end{aligned}\right\},\ \ldots\ldots$$
And we may further deduce
$$x_{2n} = 2y_n^2-\tfrac{1}{2}(y_n+y_{n-1})^2,\quad y_{2n} = y_n^2+\tfrac{1}{2}(y_n-y_{n-1})^2,$$
and thence the sides of the $2n$th triangle are

$$y_n(y_n - y_{n-1}), \quad y_n^2 - \tfrac{1}{4}(y_n - y_{n-1})^2, \quad y_n^2 + \tfrac{1}{4}(y_n - y_{n-1})^2.$$

When $n = 40$, $y_{39} = 6133864079332240379990008001809$,

$y_{40} = 3575077977948634627394046618865$;

hence the sides of the 80th triangle are found to have the values given in the Question.

[Mr. MARTIN's expressions (4) are general ones for the sides of a right-angled triangle whose legs differ by unity. See Solutions of Quest. 2981 (*Reprint*, Vol. XIV., pp. 89, 90) and Quest. 4102 (*Reprint*, Vol. XX., p. 21.]

II. *Solution by* ASHER B. EVANS, M.A.

The general solution of $u^2 + v^2 = \square$ is $u = x(a^2 - b^2)$ and $v = x(2ab)$, where a and b are any integers prime to each other, and x is either any integer, or, when $(a-b)$ is even, half any integer.

Since, by the question, $u - v = \pm 1$, and therefore

$$x\left\{(a-b)^2 - 2b^2\right\} = \pm 1,$$

we must take $x = 1$ or $\tfrac{1}{2}$; whence $(a-b)^2 - 2b^2 = \pm 1$ or ± 2.

Since the values of $(a-b)$ and b will only be interchanged by taking ± 2 instead of ± 1, all solutions of the question are included in

$$(a-b)^2 - 2b^2 = \pm 1 \quad \text{.............................. (1).}$$

The general solution of (1) is, by Lagrange's well known theorem, $(a-b) = p_n$ and $b = q_n$, where $\dfrac{p_n}{q_n}$ is the nth convergent to $\sqrt{2}$, and $(a-b)^2 - 2b^2 = +1$ and -1 alternately. The general values of p_n and q_n are shown by Lagrange to be

$$\left.\begin{aligned} p_n &= \tfrac{1}{2}\left\{(1+\sqrt{2})^n + (1-\sqrt{2})^n\right\} \\ q_n &= \frac{1}{2\sqrt{2}}\left\{(1+\sqrt{2})^n - (1-\sqrt{2})^n\right\} \end{aligned}\right\} \quad \text{.................. (2);}$$

therefore $\quad u = (a^2 - b^2) = p_n(p_n + 2q_n), \quad v = 2q_n(p_n + q_n),$

and $\quad (u^2 + v^2)^{\frac{1}{2}} = (p_n + q_n)^2 + q_n^2$ .... (3).

When $n = 80$, equations (2) will give p_{80} and q_{80}, and equations (3) will then give the sides of the 80th triangle

$(1+\sqrt{2})(1+\sqrt{2}) = (1+\sqrt{2})^2 = 3 + 2\sqrt{2}$,

$(1+\sqrt{2})^2(1+\sqrt{2})^2(1+\sqrt{2}) = (1+\sqrt{2})^5 = 41 + 29\sqrt{2}$,

$(1+\sqrt{2})^5(1+\sqrt{2})^5 = (1+\sqrt{2})^{10} = 3363 + 2378\sqrt{2}$,

$(1+\sqrt{2})^{10}(1+\sqrt{2})^{10} = (1+\sqrt{2})^{20} = 22619537 + 15994428\sqrt{2}$,

$(1+\sqrt{2})^{20}(1+\sqrt{2})^{20} = (1+\sqrt{2})^{40} = 1023286908188737$
$+ 723573111879672\sqrt{2}$,

$(1+\sqrt{2})^{40}(1+\sqrt{2})^{40} = (1+\sqrt{2})^{80} = 2094232192940929332692027310337$
$+ 1480845785007705294702019308528\sqrt{2}$.

Similarly $(1-\sqrt{2})^{80} = 2094232192940929332692027310337$
$- 1480845785007705294702019308528\sqrt{2}$;

therefore

$$p_{80} = \tfrac{1}{2}\left\{(1+\sqrt{2})^{80}+(1-\sqrt{2})^{80}\right\} = 20942321929409293326920273 10337,$$

$$q_{80} = \frac{1}{2\sqrt{2}}\left\{(1+\sqrt{2})^{80}-(1-\sqrt{2})^{80}\right\} = 14808457850077052947020193 08528,$$

$$p_{80}^2 = 438580847795017386272679123922210335156920200802026250 7053569,$$

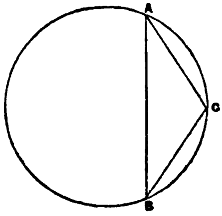
$$q_{80}^2 = 219290423897508693136339561961105167578460100401013125 3526784,$$

$$2p_{80}q_{80} = 620246983148803726504183473348903510862669060251854481 3307872,$$

and the three numbers are thus found to be the same as those given in the Question.

4135. (Proposed by J. J. Walker, M.A.)—A vertical circle may be divided into two unequal segments by a vertical chord, so that the times of sliding down any other chord in the smaller segment, drawn from an extremity of the first, with friction proportional to pressure, shall be equal to the time of falling freely down the vertical chord; and of all straight lines which can be drawn in the same vertical plane to meet a given inclined plane from a given point above it, the line of quickest descent, friction being proportional to pressure, bisects the angle between any two of equally quick descent, and the final velocity for the first will be a mean proportional between the final velocities for the latter two.

Solution by C. Leudesdorf.



1. Let AB be the vertical chord, AC another chord, making an angle α with AB. If the friction $=\mu$R, and AC be the axis of x, we have,

since $\qquad 0 = g\sin\alpha - \text{R},$

$$\frac{d^2x}{dt^2} = g\cos\alpha - \mu\text{R} = g(\cos\alpha - \mu\sin\alpha),$$

therefore $\qquad x = \tfrac{1}{2}gt^2(\cos\alpha - \mu\sin\alpha),$

since $x=0$, $dx=0$, $t=0$, together.

Hence for the time t of falling down AC,

$$\text{AC} = \tfrac{1}{2}gt^2(\cos\alpha - \mu\sin\alpha) = \tfrac{1}{2}gt^2\frac{\sin(\theta+\alpha)}{\sin\theta}, \text{ writing } \cot\theta \text{ for } -\mu;$$

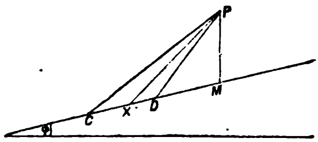
and for the time $(=t')$ of falling freely down AB,

$$\text{AB} = \tfrac{1}{2}gt'^2, \quad\text{therefore } \frac{\text{AC}}{\text{AB}} = \frac{t^2\sin(\theta+\alpha)}{t'^2\sin\theta};$$

therefore $\qquad$ if $t=t'$, $\dfrac{\sin(\theta+\alpha)}{\sin\theta} = \dfrac{\text{AC}}{\text{AB}} = \dfrac{\sin(\alpha+\text{ACB})}{\sin\text{ACB}};$

and therefore it suffices to draw AB so that the angle ACB may be equal to θ or $\cot^{-1}(-\mu)$.

2. Let PC, PD be two lines of equally quick descent, PX that of quickest descent. Let them make angles α, β, θ with the vertical.



Then $\dfrac{\cos\alpha-\mu\sin\alpha}{\cos\beta-\mu\sin\beta}=\dfrac{PC}{PD}$

$$=\frac{\cos(\beta+\phi)}{\cos(\alpha+\phi)} \;\;\ldots\ldots\ldots\; (1),$$

if $\phi=$ angle of the plane;

also, $\dfrac{2PM}{t^2}=(\cos\theta-\mu\sin\theta)\cos(\theta+\phi)$, [because $PX=PM\sec(\theta+\phi)$,]

which becomes a maximum when $\tan(2\theta+\phi)=-\mu$.

In this case, (1) gives $\dfrac{\cos(\beta+\phi)}{\cos(\alpha+\phi)}=\dfrac{\cos(2\theta+\alpha+\phi)}{\cos(2\theta+\beta+\phi)}$;

therefore $2\theta=\alpha-\beta$; that is, PX bisects the angle CPD.

If v_1, v_2, v be the final velocities at CDX respectively, we have

$$v^2_1=2g\,.\,PC(\cos\alpha-\mu\sin\alpha)=2g\,.\,\frac{PM}{\cos(\alpha+\phi)}\,.\,\frac{\cos(\beta+\phi)}{\cos(2\theta+\phi)}.$$

So $$v^2_2=2g\,.\,\frac{PM}{\cos(\beta+\phi)}\,.\,\frac{\cos(\alpha+\phi)}{\cos(2\theta+\phi)},$$

and $$v^2=2g\,\frac{PM}{\cos(\theta+\phi)}\,.\,\frac{\cos(\theta+\phi)}{\cos(2\theta+\phi)};$$

therefore $v^4=v_1^2v_2^2$; or v is a mean proportional between v_1 and v_2.

4157. (Proposed by the Rev. W. A. Whitworth, M.A.)—Two similar lumps of ice are melting, and the diminution of volume in any instant is proportional to the area of surface of each lump. Do the volumes tend to equality?

Solution by J. L. Mackenzie.

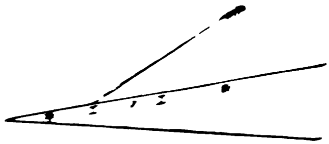
Let $\mu\,(>1)$ be the ratio of linear similarity of the two lumps, V the volume of the smaller lump, and δ an elemental diminution of V; then μ^3V is the volume of the larger, and $\mu^2\delta$ its corresponding elemental diminution. Therefore the ratio of the volumes after variation is

$$\frac{\mu^3V-\mu^2\delta}{V-\delta}=\frac{\mu^3(\mu V-\delta)}{\mu V-\mu\delta};$$

and this ratio is greater than μ^3, since $\mu V-\delta>\mu V-\mu\delta$. Therefore the volumes tend to greater inequality.

In the same manner it may be shown generally, that if μx and x be two variable magnitudes ($\mu>1$), and h and k small increments of μx and x; then if h and k are both positive, the magnitudes μx and x are tending to equality or to greater inequality according as $\dfrac{h}{k}$ is greater or less than μ.

2. Let PC, PD be two lines of equally quick descent, PX that of quickest descent. Let them make angles α, β, θ with the vertical.



Then $\dfrac{\cos\alpha - \mu\sin\alpha}{\cos\beta - \mu\sin\beta} = \dfrac{P[illegible]}{P[illegible]}$

$= \dfrac{\cos(\beta-\phi)}{\cos(\alpha-\phi)}$ .... [illegible]

if ϕ = angle of the plane.

also, $\dfrac{2PM}{g} = \cos\theta - \mu\sin\theta$ [illegible] $(\theta-\phi)$, [illegible] $= PX$ [illegible] $(\theta-\phi)$,

which becomes a maximum when [illegible]

In this case [illegible] gives [illegible]

therefore $2\theta = \alpha - \beta$ [illegible]

If [illegible]

[illegible]

So [illegible]

and [illegible]

therefore [illegible]

4157. [illegible]

[illegible]

Solution by [illegible]

Let μ (>1) be the ratio of [illegible] volume of the smaller [illegible] μ^3V is the volume of the larger [illegible] nution. Therefore the ratio of [illegible]

$$\frac{\mu^3V - [illegible]}{V - [illegible]} [illegible]$$

and this ratio is greater than [illegible] volumes tend to greater inequality [illegible]

In the same manner it may [illegible] variable magnitudes ($\mu >$ [illegible] then if h and k are both positive [illegible] equality or to greater inequality [illegible]

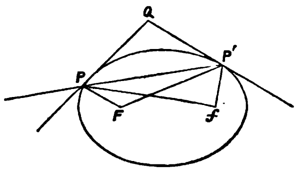
For the ratio of the magnitudes after variation is

$$\frac{\mu x + h}{x+k} = \frac{\mu(\mu x + h)}{\mu x + \mu k};$$

and when h and k are positive this ratio is $>, =, < \mu$ according as $h >, =, < \mu k$, that is, as $\frac{h}{k} >, =, < \mu$. And *vice versâ* if h and k are negative.

3936. (Proposed by T. T. WILKINSON, F.R.A.S.)—Prove that, in an ellipse, the sum of the angles subtended at the foci by any chord, together with twice the angle which the chord subtends at its pole, are equal to four right angles.

Solution by the PROPOSER; A. B. EVANS, M.A.; *and others.*



Let F, f be the foci of the ellipse; P, P′ the polar of Q; and join FP, FP′; also fP, fP′. Put $\angle f\mathrm{P'F} = \alpha$; $\angle f\mathrm{PF} = \beta$; and let any right angle equal R. Then, taking the quadrilateral fP′QP, we have (Besant's *Conics*, Prop. lx. p. 55) $\angle f\mathrm{P'Q} = \mathrm{R} + \frac{1}{2}\alpha$, $\angle f\mathrm{PQ} = \mathrm{R} - \frac{1}{2}\beta$.

Hence $\angle f\mathrm{P'Q} + \angle \mathrm{P'QP} + \angle f\mathrm{PQ} + \angle \mathrm{P}f\mathrm{P'}$

$$= \angle \mathrm{P'QP} + \angle \mathrm{P}f\mathrm{P'} + (\mathrm{R} + \tfrac{1}{2}\alpha + (\mathrm{R} - \tfrac{1}{2}\beta) = 4\mathrm{R};$$

therefore $$\angle \mathrm{P'QP} + \angle \mathrm{P}f\mathrm{P'} + \tfrac{1}{2}(\alpha - \beta) = 2\mathrm{R} \quad\ldots\ldots\ldots (1).$$

Similarly, from the quadrilateral FPQP′, since

$$\angle \mathrm{FP'Q} = \mathrm{R} - \tfrac{1}{2}\alpha, \text{ and } \angle \mathrm{FPQ} = \mathrm{R} + \tfrac{1}{2}\beta,$$

we have $$\angle \mathrm{P'QP} + \angle \mathrm{P'FP} + \tfrac{1}{2}(\beta - \alpha) = 2\mathrm{R} \quad\ldots\ldots\ldots (2).$$

Hence from (1) and (2) we finally obtain

$$\angle \mathrm{P'FP} + \angle \mathrm{P}f\mathrm{P'} + 2\angle \mathrm{P'QP} = 4\mathrm{R}.$$

[This is really the same theorem as "The external angle between two tangents to an ellipse is equal to the semi-sum of the angles subtended by the chord of contact at the foci;" and this proposition Mr. Wolstenholme states that he set in the Senate House in 1862, but which was given to him as a new theorem by Mr. B. W. Horne, Fellow and Lecturer at St. John's College. It is now given as an ordinary proposition in several of the numerous books on Geometrical Conics.]

4036. (Proposed by the EDITOR.)—Let $\rho_1, \rho_2, \rho_3, \mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3$ be the radii of the circles drawn in and about the three triangles cut off from the corners of a given triangle by tangents to its inscribed circle drawn parallel

to the opposite sides; then, adopting the usual notation for the other lines connected with the triangle ($s_1 = s - a,\ s_2 = s - b,\ s_3 = s - c$), and putting

$$r \tan \tfrac{1}{2}\mathrm{A} \tan \tfrac{1}{2}\mathrm{B} \tan \tfrac{1}{2}\mathrm{C} = \frac{r^2}{s} = \lambda, \text{ and } \frac{\Sigma (s_1^2)}{s^2} = \frac{a^2+b^2+c^2-s^2}{s^2} = \mu,$$

prove that $\rho_1 + \rho_2 + \rho_3 = r,\quad r_1\rho_1 = r_2\rho_2 = r_3\rho_3 = r^2,\quad \rho_1\rho_2\rho_3 = \lambda^2 r = \dfrac{r^2}{s^2}r^3;$

$$\frac{\rho_1}{s_1} = \frac{\rho_2}{s_2} = \frac{\rho_3}{s_3} = \frac{r}{s} = \frac{\lambda}{r} = \frac{\rho_2+\rho_3}{a} = \frac{\rho_3+\rho_1}{b} = \frac{\rho_1+\rho_2}{c} = \frac{(\rho_1\rho_2\rho_3)^{\frac13}}{(\rho_1+\rho_2+\rho_3)^{\frac13}};$$

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = \frac{\Sigma(s_2s_3)}{r^3} = \frac{as_1 + s_2s_3}{r^3} = \frac{2(bc+ca+ab)-(a^2+b^2+c^2)}{4r^3};$$

$$r_1\left(\frac{1}{\rho_2} - \frac{1}{\rho_3}\right) + r_2\left(\frac{1}{\rho_3} - \frac{1}{\rho_1}\right) + r_3\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = 0;$$

$$\rho_2\rho_3 + \rho_3\rho_1 + \rho_1\rho_2 = \frac{r^2}{s^2}\Sigma(s_2s_3) = \frac{r^2}{2s^2}\left\{2s^2 - (a^2+b^2+c^2)\right\};$$

$$\rho_1^2 + \rho_2^2 + \rho_3^2 = \frac{r^2}{s^2}(a^2+b^2+c^2-s^2) = \mu\frac{r^2}{s^2};\quad r_1\rho_1^2 + r_2\rho_2^2 + r_3\rho_3^2 = r^3;$$

$$\Sigma(\mathrm{R}_1) = \mathrm{R};\ \Sigma(\mathrm{R}_2\mathrm{R}_3) = \frac{\mathrm{R}^2}{s^2}\Sigma(s_2s_3) = \frac{\mathrm{R}^2}{s^2}(as_1 + s_2s_3);$$

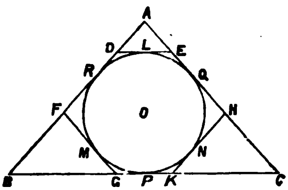
$$\Sigma(\mathrm{R}_1^2) = \mu\frac{\mathrm{R}^2}{s^2};\quad \mathrm{R}_1\mathrm{R}_2\mathrm{R}_3 = \frac{r^2}{s^2}\mathrm{R}^3.$$

If, moreover, a second series of circles be obtained by drawing tangents similarly to the first series (ρ_1, ρ_2, ρ_3), and inscribing circles in the triangles thus cut off; and then a third series, and so on *ad infinitum;* prove that the sum of the radii of the nth series of circles is likewise equal to the radius (r) of the inscribed circle of the triangle ABC, that the sum of the areas of the same series of circles is μ^n times the area of the inscribed circle, and that the sum of the areas of the infinite series of such circles is

$$\frac{\pi r^2}{1-\mu} = \frac{\pi\Delta^2}{2s^2-(a^2+b^2+c^2)} = \frac{2\pi\Delta^2}{2(bc+ca+ab)-(a^2+b^2+c^2)}.$$

Also deduce some of the many other relations amongst these magnitudes.

I. *Solution by the* Rev. J. L. KITCHIN, M.A.; A. RENSHAW; *and others.*



The triangle ADE is similar to ABC, hence if p_1 be the perpendicular from A on BC, we have

$$\frac{p_1 - 2r}{p_1} = \frac{\mathrm{DE}}{\mathrm{BC}},$$

$$\therefore\quad \mathrm{DE} = \left(1 - \frac{2r}{p_1}\right)a$$

$$= \left(1 - \frac{2\Delta}{s}\cdot\frac{a}{2\Delta}\right)a = \frac{as_1}{s};$$

but $\dfrac{\rho_1}{r} = \dfrac{\mathrm{DE}}{\mathrm{BC}} = \dfrac{s_1}{s}$; therefore $\dfrac{r}{s} = \dfrac{\rho_1}{s_1} = \dfrac{\rho_2}{s_2} = \dfrac{\rho_3}{s_3} = \dfrac{\rho_1+\rho_2+\rho_3}{s_1+s_2+s_3}$ ....(1).

Now $s_1 + s_2 + s_3 = s$, therefore, by (1), $\rho_1 + \rho_2 + \rho_3 = r$ ....(2).

Again $\quad r_1 = \dfrac{\Delta}{s_1} = \dfrac{rs}{s_1} = \dfrac{r^2}{\rho_1}$, by (1); $\quad r_2 = \dfrac{r^2}{\rho_2}$, $\quad r_3 = \dfrac{r^2}{\rho_3}$;

therefore $$\rho_1 r_1 = \rho_2 r_2 = \rho_3 r_3 = r^2 \quad\text{.......................... (3).}$$

Also $$\lambda = r \tan \tfrac12 A \tan \tfrac12 B \tan \tfrac12 C = r\,\frac{(s s_1 s_2 s_3)^{\frac12}}{s^2} = \frac{r\Delta}{s^2} = \frac{r^2}{s},$$

therefore, by (1), $$\rho_1\rho_2\rho_3 = \frac{r^3 s_1 s_2 s_3}{s^3} = \frac{r^3\Delta^2}{s^4} = \frac{r^5}{s^2} = \lambda^2 r \quad\text{.................. (4).}$$

From (4), $\dfrac{\lambda}{r} = \dfrac{r}{s}$, therefore $$\frac{\rho_1}{s_1} = \frac{\rho_2}{s_2} = \frac{\rho_3}{s_3} = \frac{r}{s} = \frac{\lambda}{r} \quad\text{........................ (5).}$$

From (5), $$\frac{r}{s} = \frac{\rho_1+\rho_2}{s_1-s_2} = \frac{\rho_1+\rho_2}{c} = \frac{\rho_1+\rho_3}{b} = \frac{\rho_2+\rho_3}{a} \quad\text{................ (6).}$$

From (4) and (2), $$\left\{\frac{\rho_1\rho_2\rho_3}{(\rho_1+\rho_2+\rho_2)^3}\right\}^{\frac13} = \left(\frac{\lambda^2 r}{r^3}\right)^{\frac13} = \frac{\lambda}{r} \quad\text{..................... (7).}$$

From (3) we get $\quad r_1\rho_1^2 + r_2\rho_2^2 + r_3\rho_3^2 = r^2(\rho_1+\rho_2+\rho_3) = r^3 \quad$ .... (8).

Similarly we find

$$\begin{aligned} r^3 &= r_1\rho_1\rho_3 + r_2\rho_1\rho_2 + r_3\rho_2\rho_3 = r_1\rho_1\rho_2 + r_2\rho_2\rho_3 + r_3\rho_1\rho_3 \\ &= (r_1r_2r_3\rho_1\rho_2\rho_3)^{\frac13} = \left\{\tfrac13(r_1\rho_1 + r_2\rho_2 + r_3\rho_3)\right\}^{\frac32} \\ &= \frac{\lambda^3 abc}{(\rho_1+\rho_2)(\rho_1+\rho_3)(\rho_2+\rho_3)}, \text{ by (6), } = \frac{abc\lambda^3}{(r-\rho_1)(r-\rho_2)(r-\rho_3)} \quad\text{...... (9),} \end{aligned}$$

Again, from (5), $$\frac{\rho_1-\rho_2}{s_1-s_2} = \frac{\rho_2-\rho_3}{s_2-s_3} = \frac{\rho_3-\rho_1}{s_3-s_1} = \frac{r}{s} = \frac{\lambda}{r} = \text{\&c.},$$

or $$\frac{\rho_1-\rho_2}{b-a} = \frac{\rho_2-\rho_3}{c-b} = \frac{\rho_3-\rho_1}{a-c} = \frac{r}{s} = \frac{\lambda}{r} \quad\text{..................... (10).}$$

This shows that $\rho_1-\rho_2$ must have the same sign as $b-a$, &c.; therefore if a, b, c are in ascending order of magnitude, then ρ_1, ρ_2, ρ_3 are in descending order of magnitude.

The cubic $\quad rr_1\rho^3 - r^2r_1\rho^2 + \lambda(ar^2 + \lambda r_1^2)\rho - \lambda^2r^2r_1 = 0 \quad$ .... (11)
gives the radii of the circles in the triangles ρ_1, &c.

From (4) and (9), we obtain

$$\frac{r^2}{\lambda^2} = \frac{r^3}{\lambda^2 r} = \frac{r^3}{\rho_1\rho_2\rho_3} = \frac{r_1}{\rho_2} + \frac{r_2}{\rho_3} + \frac{r_3}{\rho_1} = \frac{r_1}{\rho_3} + \frac{r_2}{\rho_1} + \frac{r_3}{\rho_2};$$

therefore $$r_1\left(\frac{1}{\rho_2} - \frac{1}{\rho_3}\right) + r_2\left(\frac{1}{\rho_3} - \frac{1}{\rho_1}\right) + r_3\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = 0 \quad\text{...... (12).}$$

The circumscribed circles would lead to many interesting relations, which may be investigated in the same way.

II. *Solution by* Asher B. Evans, M.A.

The perimeter of the triangle ADE is evidently $2s_1$, since AD + DL = AR and AE + EL = AQ. Similarly the perimeters of BFG and CHK are $2s_2$ and $2s_3$ respectively. Again, since the four triangles ADE, BFG, CHK, ABC are similar, we have $$\frac{\rho_1}{s_1} = \frac{\rho_2}{s_2} = \frac{\rho_3}{s_3} = \frac{r}{s} \quad\text{.............................. (1),}$$

therefore $$\frac{\rho_1+\rho_2+\rho_3}{s_1+s_2+s_3} = \frac{r}{s}; \text{ but } s_1+s_2+s_3 = s,$$

therefore $$\rho_1+\rho_2+\rho_3 = r \quad\text{.......}\quad (2).$$

For the area of the triangle ABC we have

$$r_1s_1 = r_2s_2 = r_3s_3 = rs \quad\text{.......}\quad (3).$$

By multiplying (1) by (3), member by member, we obtain

$$r_1\rho_1 = r_2\rho_2 = r_3\rho_3 = r^2 \quad\text{.......}\quad (4).$$

From (1), we have

$$\rho_2+\rho_3 = \frac{r}{s}(s_2+s_3) = \frac{ra}{s}, \quad \rho_3+\rho_1 = \frac{rb}{s}, \quad \rho_1+\rho_2 = \frac{rc}{s},$$

therefore $$\frac{r}{s} = \frac{\rho_2+\rho_3}{a} = \frac{\rho_3+\rho_1}{b} = \frac{\rho_1+\rho_2}{c} \quad\text{.......}\quad (5).$$

Again, $$\tan^2\tfrac{1}{2}A = \frac{s_2s_3}{ss_1} = \frac{r^2}{s_1^2}, \quad \tan^2\tfrac{1}{2}B = \frac{s_1s_3}{ss_2} = \frac{r^2}{s_2^2}, \quad \tan^2\tfrac{1}{2}C = \frac{s_1s_2}{ss_3} = \frac{r^2}{s_3^2};$$

therefore $$s_1s_2s_3 = sr^2 \text{ and } \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B \tan\tfrac{1}{2}C = \frac{r^3}{s_1s_2s_3} \quad\text{.......}\quad (6);$$

hence $$\lambda = r\tan\tfrac{1}{2}A \tan\tfrac{1}{2}B \tan\tfrac{1}{2}C = \frac{r^4}{s_1s_2s_3} = \frac{r^2}{s} \quad\text{.......}\quad (7).$$

From (1) and (6), we have $$\rho_1\rho_2\rho_3 = \frac{r^5}{s^2} \quad\text{.......}\quad (8);$$

from (7) and (8), we have $$\rho_1\rho_2\rho_3 = \lambda^2 r \quad\text{.......}\quad (9);$$

from (2) and (8), we have $$\frac{(\rho_1\rho_2\rho_3)^{\frac{1}{2}}}{(\rho_1+\rho_2+\rho_3)^{\frac{3}{2}}} = \frac{r}{s} = \frac{\lambda}{r} \quad\text{.......}\quad (10).$$

III. *Solution by the* Proposer.

By well known properties, we have

$$\rho_1(\cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C) = DE = r(\tan\tfrac{1}{2}B + \tan\tfrac{1}{2}C)$$
$$= r(\cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C)\tan\tfrac{1}{2}B \tan\tfrac{1}{2}C,$$

therefore $$\rho_1\tan\tfrac{1}{2}A = \rho_2\tan\tfrac{1}{2}B = \rho_3\tan\tfrac{1}{2}C = \lambda = \frac{r^2}{s};$$

but $$\tan\tfrac{1}{2}A = \frac{r}{s_1}, \text{ \&c.; therefore } \frac{\rho_1}{s_1} = \frac{\rho_2}{s_2} = \frac{\rho_3}{s_3} = \frac{r}{s}.$$

We thus see that the triangles ADE, BFG, CKH, ABC are similar, and have their corresponding lines in the ratios

$$s_1 : s_2 : s_3 : s, \text{ where } s_1+s_2+s_3 = s;$$

whence it follows that any line connected with the triangle ABC is equal to the sum of the three similar lines of the three corner triangles.

We have thus, for instance,

$$\Sigma(\rho_1) = r; \quad \Sigma(R_1) = R; \quad \Sigma(\rho_1^2) = \frac{r^2}{s^2}\Sigma(s_1^2) = \mu r^2;$$

$$\Sigma(R_1^2) = \frac{R^2}{s^2}\Sigma(s_1^2) = \mu R^2; \quad \Sigma(\rho_2\rho_3) = \frac{r^2}{s^2}\Sigma(s_2s_3), \text{ \&c.};$$

and the other properties given in the first part of the question, as well as many similar ones, may be proved in like manner.

Moreover, in any series of the circles obtained as described in the last part of the question, the radii of any triad are together equal to the radius of the circle around which they stand; hence the sum of the radii of the whole of the nth series must be equal to r.

Again, since $\Sigma(\rho_1^2) = \mu r^2$, the areas of the three circles (ρ_1), (ρ_2), (ρ_3) are together equal to μ times the area of the circle (r); also, since the triangles cut off are all similar to ABC, μ is a constant throughout the whole series of circles; thus the sum of the areas of the second series of circles must be μ times that of the first series, or μ^2 times the area of the circle (r); and so on *ad infinitum*. Hence the sum of the areas of the infinite series of such circles is

$$(1+\mu+\mu^2+\ldots)\,\pi r^2 = \frac{\pi r^2}{1-\mu} = \frac{\pi r^2 s^2}{2s^2-(a^2+b^2+c^2)},$$

which may be expressed in either of the forms given in the question.

The expression for $\Sigma(\rho_2\rho_3)$ given in the cubic (11) of the first solution may be readily shown to agree with that given in the question; for

$$\frac{\lambda}{rr_1}(ar^2+\lambda r_1^2) = \frac{r^2}{s^2}\left(\frac{ars}{r_1}+rr_1\right) = \frac{r^2}{s^2}(as_1+s_2s_3).$$

Many other relations amongst these magnitudes may be readily investigated.

4069. (Proposed by Professor CLIFFORD.)—1. Curves of order $2n+1$ pass n times through each circular point, and through n^2+4n+1 other fixed single points (or their equivalent in multiple points); show that the envelope of their asymptotes is a tricusped hypocycloid.

2. Curves of order $2n+2$ pass n times through each circular point and through n^2+6n+4 other fixed points, and their real asymptotes are at right angles; show that the envelope of their asymptotes is a tricusped hypocycloid.

Solution by the PROPOSER.

Professor WOLSTENHOLME's remark in his solution of this question, given on p. 31 of this volume of the *Reprint*, that the first part "is not quite true as it stands," has led me to examine the whole with the help of his method; and it turns out, singularly enough, that it is the *second* part that requires correction, not the first. The way in which this comes about is instructive, and the corrected theorem leads us to consider a somewhat interesting series of curves.

1. I will first state the grounds on which I originally concluded that these theorems were true. It is required to find the envelope of the asymptotes of a pencil of curves which if of odd order have *one* real point at infinity besides the circular points, if of even order *two* which are at right angles or harmonic of the circular points. The intersections with the line infinity at the circular points are due to multiplicity of these points, not to contact with the line infinity.

Now, first, *the line infinity is a tangent to this envelope at each of the cir-*

cular points and no otherwhere. For the line infinity can only become an asymptote by the variable one point or one of the variable two points at infinity coming to coincide with one of the circular points. In the second case the variable two points being harmonic of the circular points, if one of them coincide with a circular point, the other must coincide with it. In both cases, then, there are two curves of the pencil which have the line infinity for asymptote; and it is clear that the intersection of the line infinity with the next consecutive asymptote (*i.e.*, its point of contact with the envelope) is the circular point at which it is an asymptote.

Next, from any point at infinity not a circular point, one tangent distinct from the line infinity can be drawn to the envelope. For there is one curve of the pencil that passes through this point.

If, then, the line infinity is an ordinary tangent at each of the circular points, we see that from any point at infinity three tangents may be drawn to the envelope; viz., the line infinity counting twice, and one other. The envelope is therefore of the third class, having the line infinity for double tangent whose points of contact are the circular points; that is to say, a hypocycloid of three branches.

In fact, the tangential equation of the curve may be at once written down. Let $i=0, j=0$ be the equations to the circular points, $k=0$ that to some other point; then the equation is $ijk+(i,j)^3=0$. It is, in fact, of the same form as the equation of a cubic curve having a node at the origin to which the axes are tangents. If for k we write $k+\lambda i+\mu j$, it is clear that by proper choice of λ, μ we can get rid of the two middle terms of $(i,j)^3$; the equation then becomes $ijk+\alpha i^3+\beta j^3=0$, which is the same as $p=a\cos 3\theta$, where p is the distance of a tangent from the origin $k=0$, and θ the angle it makes with a fixed line. (SALMON, *Higher Plane Curves*, p. 271, Ex. 5.)

This result is true *if the line infinity is an ordinary tangent at each of the circular points.* Now this holds good in the *first case* of the question; for in this the one variable point at infinity is made to move up to a multiple point, and so only one branch acquires an ordinary contact; in virtue of this, then, the line infinity counts only once as an asymptote for each circular point. It also holds good in the already well-known case of curves of the second order, *i. e.*, in the second case of the question when $n=0$. For in this only the two variable points at infinity coincide at a circular point, making again an ordinary contact.

But in the second case of the question, when n is not zero, something different happens. Here the two variable points at infinity simultaneously approach a circular point which is already multiple on the curve; they approach it on the same branch, and *produce a point of inflexion on that branch.* In respect of each circular point, therefore, the line infinity counts for *two* asymptotes; the envelope is raised to the *fifth* class, and *has the line infinity for inflexional tangent at each circular point.*

2. This synthetic discussion shall now be confirmed by analysis. In the first case, the equation of a curve of the pencil is

$$(x+\lambda y)(x^2+y^2)^n+k(\lambda,1\between x,y)^2.(x^2+y^2)^{n-1}+\ldots=0,$$

and its real asymptote is

$$(x+\lambda y)(1+\lambda^2)+k(\lambda,1\between\lambda,-1)^2=0,$$

whose envelope is of the third class, touched by $k=0$ (the line infinity) for the two values $\lambda=\pm(-1)^{\frac{1}{2}}$; whence as before.

In the second case, it is convenient to write the equation of the variable curve in the form

$$\left\{x^2 + \left(\lambda - \frac{1}{\lambda}\right) xy - y^2\right\} (x^2+y^2)^n + k\left(\lambda - \frac{1}{\lambda}, 1\right)\!\left(x, y\right)^3 . (x^2+y^2)^{n-1} + \ldots = 0.$$

The two real asymptotes are

$$\left(x - \frac{y}{\lambda}\right)(1+\lambda^2)^2 + k\left(\lambda - \frac{1}{\lambda}, 1\right)\!\left(1, \lambda\right)^3 = 0,$$

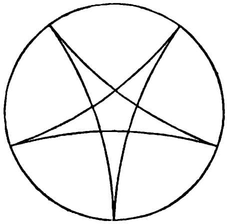
$$(x+\lambda y)(1+\lambda^2)^2 + \lambda k\left(\lambda - \frac{1}{\lambda}, 1\right)\!\left(\lambda, -1\right)^3 = 0.$$

These have the same envelope, as one equation is got from the other by writing $-\lambda^{-1}$ for λ. The envelope is of the fifth class, touched by $k=0$ twice for each of the values $\lambda = \pm(-1)^{\frac{1}{2}}$. The line infinity is therefore a double tangent with united contacts (*i. e.*, an inflexional tangent; just as a cusp is a double point with united branches) at each of the circular points.

3. It remains to investigate the nature of a curve of the fifth class having the line infinity for inflexional tangent at each of the circular points. This singularity being equivalent to two inflexions and four double tangents, Plücker's equations at once tell us that the curve is of the sixth order and has five cusps and five nodes. Its tangential equation may be at once written down, being of the same form as that of a quintic curve having a quadruple point at the origin, two of whose branches coincide with each of the axes; namely, it is $i^2j^2k + (i, j)^5 = 0$, where $i=0$, $j=0$ are the circular points, and $k=0$ is some other point. As before, we may suppose k to have been so selected as to get rid of the two middle terms of $(i, j)^5$. Now a particular case of the equation is

$$i^2j^2k + \alpha i^5 + \beta j^5 = 0, \text{ or } p = a \cos 5\theta,$$

which represents the hypocycloid (Fig. 1) described by a point on a rolling circle whose radius is two-fifths of the radius of the fixed circle.



(Fig. 1.)

The general equation may be transformed into $\quad p = a \cos 5\theta + b \cos 3\theta + c \sin 3\theta,$

or, if we write

$$p_1 = 2a \cos 5\theta, \quad p_2 = 2b \cos 3\theta + 2c \sin 3\theta,$$

the equation is $\quad 2p = p_1 + p_2.$

Now p_2 and p_1 are the distances from the origin of parallel tangents to a three-cusped and a sextic five-cusped hypocycloid respectively; whence we learn that *the curve in question is the envelope of a line midway between parallel tangents to two such hypocycloids.*

These hypocycloids have only to be concentric; and their relative size and orientation are the two variable elements in the equation of our curve. The method of description by tangents, however, gives us immediately a description by points; since it is clear that the point of contact of the variable tangent bisects the line joining the points of contact of the two tangents to which it is parallel and intermediate. In this way it is easy to draw roughly a few typical forms.

4. A hypocycloid in which the radii of the rolling and fixed circles are to one another as n to $2n+1$ is a curve of order $2n+2$, class $2n+1$, with $2n+1$ cusps, $(n-1)(2n+1)$ nodes, and has $(n+1)$-pointic contact with the line infinity at each circular point. Its tangential equation is

$$i^n j^n k + \alpha i^{2n+1} + \beta j^{2n+1} = 0;$$

or, which is the same thing, $\quad p = a \cos (2n+1)\theta.$

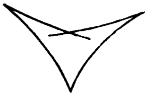
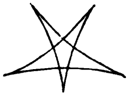
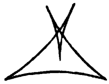
To this simplest class of roulettes, all whose tangential singularities are at infinity, it may be permissible to give the name "stars." Thus an ordinary tricusp is a three-rayed star, the curve in Fig. 1 is a five-rayed star, and so on. We may now state the following proposition:—

Every curve of class $2n+1$, which has $(n+1)$-pointic contact with the line infinity at each circular point, is the envelope of a line which is the mean of the parallel tangents of n concentric stars of all odd classes up to $2n+1$.

Namely, its equation is $\quad i^n j^n k + (i, j)^{2n+1}$,

or $$p = a_3 \cos 3\theta + b_3 \sin 3\theta + \ldots + a_{2n+1} \cos (2n+1)\,\theta,$$

from which the proposition is obvious. The curve has the same number of nodes and cusps as a star of class $2n+1$; only they need not, as in the case of the star, be all real.


(Fig. 2.) (Fig. 3.) (Fig. 4.) (Fig. 5.) (Fig. 6.)

I remark in conclusion, first, that the point-equation of our curve of the fifth class is $\quad \text{Disct.}\,(a,\, b,\, x+iy,\, x-iy,\, e,\, f \between \lambda,\, \mu)^5 = 0,$

which is worked out in Dr. SALMON'S *Higher Algebra*, and secondly, that the Hessian of the tangential equation is easily calculated and shows the cuspidal tangents to be common tangents of a three- and a five-rayed star.

4168. (Proposed by Sir JAMES COCKLE, F.R.S.)—Boole, apparently following a statement of Lacroix, intimates (pp. 89, 90, Ex. 7, 8) that for each of the forms $$\frac{dy}{dx} + y^2 \pm \left(\frac{dP}{dx} + P^2\right)$$ there may be deduced a similar expression for an integrating factor. Test the statement.

Solution by the PROPOSER.

1. If we take the negative sign, the factor $\dfrac{e^{-2\int P\,dx}}{(y-P)^2}$ reduces the form to
$$\frac{d}{dx}\left\{x - \frac{e^{-2\int P\,dx}}{y-P}\right\}.$$
But no similar factor is indicated for the positive sign, y^2+P^2 not being divisible by $y+P$.

2. The more convenient factor $e^{\int (y+P)\,dx}$ reduces the form to
$$e^{\int P\,dx} \frac{d^2}{dx^2}\left(e^{\int y\,dx}\right) \pm e^{\int y\,dx} \frac{d^2}{dx^2}\left(e^{\int P\,dx}\right),$$
and when the negative sign is taken this becomes

$$\frac{d}{dx}\left\{e^{\int \mathrm{P}\,dx}\frac{d\,.\,e^{\int y\,dx}}{dx}-e^{\int y\,dx}\frac{d\,.\,e^{\int \mathrm{P}\,dx}}{dx}\right\}.$$

But no similar result is indicated for the positive sign.

3. I take this opportunity of pointing out that in Exs. 5 and 6 of p. 458 of Boole, "a" is wrongly inserted in place of "x." The like error occurs at p. 181 of Vol. V. of the *Cambridge and Dublin Mathematical Journal*, where "a" is wrongly inserted in place of "x."

Note on Pythagorean Triangles (Question 4102).

By J. W. L. Glaisher, B.A.

The inferences which Mr. H. S. Monck has drawn (see pp. 20, 21 of this volume) from Mr. Wilkinson's very elegant theorem with regard to the formation of right-angled triangles whose sides and hypotenuses are integers (Pythagorean triangles, as they are frequently called) are not correct.

Mr. Wilkinson's theorem, practically, is that if $a^2+b^2=c^2$, and

$$a'=2a+2c+b,\quad c'=2a+3c+2b,\quad b'=a+2c+2b,$$

then $a'^2+b'^2=c'^2$ and $a'-b'=a-b$; by means of this theorem it follows that if any Pythagorean triangle (the difference of the sides being m) is given, an infinite series of Pythagorean triangles, also having their sides differing by the same quantity, can be derived in the simple manner indicated by Mr. Wilkinson. But Mr. Monck's first result, viz., that all the Pythagorean triangles having the same side-difference can be obtained in this way from any one of them, is certainly not proved by him; and the second, that all Pythagorean triangles of side-difference m have their sides multiples of Pythagorean triangles of side-difference unity, is not true (*e. g.*, 5, 12, 13 form a Pythagorean triangle). All that follows directly from the reasoning is that any Pythagorean triangle being given, another such triangle can be found with greater, and another with (algebraically) less, sides; but it does not follow necessarily that we thus get, as Mr. Monck assumes, always to the triangle 1, 1, 0. It is interesting to observe how, when a triangle with a negative side is obtained, we pass in effect to a triangle with a different side-difference, thus from 5, 13, 12 we pass to -4, 5, 3, viz., 4, 5, 3, with a side-difference unity. The proposition seems to suggest several questions worth investigation with regard to this transition, and other matters.

I may remark that a method of writing down all the Pythagorean triangles is given by Mr. Sang, in the *Edinburgh Transactions* for 1864 (vol. xxiii. p. 757). Form a list of the primes $\equiv 1$ (mod. 4) and decompose each (as can always be done) into the sum of two squares, so that $p^2+q^2=\gamma$, then $a=2pq$, $b=p^2-q^2$, the sides of the triangles of which the prime γ is the hypotenuse; having thus constructed the table of triangles with prime hypotenuses, we can obtain, by means of a few simple subsidiary theorems, the triangles with composite hypotenuses. Mr. Sang, at the end of his Memoir, gives a list of Pythagorean triangles, arranged according to hypotenuses up to 1105, 1073, 264. A similar table is also given in the second volume of Schulze's *Sammlung logarithmischer Tafeln*, Berlin, 1778, vol. ii.

3115. (Proposed by Dr. S. HART.)—To find five biquadrate numbers whose sum is a biquadrate number.

Solution by ARTEMAS MARTIN *and* Rev. U. JESSE KNISELY.

Let x, $x+2a$, $x+4a$, $x+5a$, $x+10a$ be the roots of the numbers, and $x+11a$ the root of the sum of the biquadrates.

Then $x^4+(x+2a)^4+(x+4a)^4+(x+5a)^4+(x+10a)^4=(x+11a)^4$.

Expanding and uniting terms, and then putting $x=aw$, we have

$$w^4+10w^3+36w^2-134w-936=0.$$

Multiplying by 16, and then putting $2w=n-5$,

$$n^4-6n^2-1512n=7891=607\times 13=13\,(756-13^2);$$

or $$n^4-6n^2+9=7900+1512n.$$

Extracting square root, &c., we have

$$n^2=3+(7900+1512n)^{\frac{1}{2}}=169-\frac{166-(7900+1512n)^{\frac{1}{2}}}{1}$$

$$=169-\frac{19656-1512n}{166+(7900+1512n)^{\frac{1}{2}}};$$

or $$n^2-\frac{1512n}{166+(7900+1512n)^{\frac{1}{2}}}=169-\frac{19656}{166+(7900+1512n)^{\frac{1}{2}}};$$

therefore $$n-\frac{756}{166+(7900+1512n)^{\frac{1}{2}}}=\pm\left(13-\frac{756}{166+(7900+1512n)^{\frac{1}{2}}}\right).$$

Taking $+$, we have $n=13=2w+5$; whence $w=4$, and $x=4a$.
If $a=1$, we have $4^4+6^4+8^4+9^4+14^4=15^4$.

4180. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—If α, β, γ be the three roots of the equation $x^3+px+q=0$, prove that

$$3\,(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5)=5\,(\alpha^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4).$$

I. *Solution by the* Rev. ROBERT BRUCE, M.A.

By known relations we have

$$\Sigma(\alpha)=0,\quad \Sigma(\beta\gamma)=p \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1, 2),$$

$$\alpha^3+p\alpha+q=0,\quad \beta^3+p\beta+q=0,\quad \gamma^3+p\gamma+q=0 \ldots\ldots (3, 4, 5).$$

$(1)^2-2\,(2)$ gives $\quad \Sigma(\alpha^2)=-2p \;\ldots\ldots\ldots\ldots\ldots (6).$

$(3)+(4)+(5)$ gives, by aid of (1), $\quad \Sigma(\alpha^3)=-3q \;\ldots\ldots\ldots\ldots\ldots (7).$

$\alpha\,(3)+\beta\,(4)+\gamma\,(5)$ gives, by (1), $\quad \Sigma(\alpha^4)=-p\Sigma(\alpha^2) \;\ldots\ldots\ldots (8).$

$\alpha^2\,(3)+\beta^2\,(4)+\gamma^2\,(5)$ gives, by (6) and (7), $\quad \Sigma(\alpha^5)=5pq \;\ldots\ldots\ldots\ldots\ldots\ldots (9).$

$5\,(7)\times(8)$ gives $\quad 5\Sigma(\alpha^3)\,\Sigma(\alpha^4)=15pq\,\Sigma(\alpha^2)=3\Sigma(\alpha^2)\,\Sigma(\alpha^5)$, by aid of (9).

II. *Solution by* JOHN C. MALET, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

Since the only combination of the coefficients of the equation $x^3+px+q=0$, which will give a symmetric function of the roots of the 7th degree, must be of the form Ap^2q, we have at once

$$(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5) = B(\alpha^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4).$$

To determine B, consider the cubic $(x+1)^2(x-2)=0$, and we have at once $\quad 6\times 30 = B\times 6\times 18$, or $B=\frac{5}{3}$;

therefore $\quad 3(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5) = 5(\alpha^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4).$

III. *Solution by* R. W. GENESE, B.A.; H. MURPHY; *and others.*

Let $\quad f(x) = (x-\alpha)(x-\beta)(x-\gamma)\ldots\ldots,$

then $\quad \log f(x) = \log(x-\alpha)+\log(x-\beta)+\ldots\ldots;$

therefore $\quad \dfrac{f'(x)}{f(x)} = \dfrac{1}{x-\alpha}+\dfrac{1}{x-\beta}+\dfrac{1}{x-\gamma}+\ldots\ldots$

$$= \frac{1}{x}\left(1+\frac{\alpha}{x}+\frac{\alpha^2}{x^2}+\frac{\alpha^3}{x^3}+\&c.\right)+\frac{1}{x}\left(1+\frac{\beta}{x}+\frac{\beta^2}{x^2}+\&c.\right)+\&c.;$$

therefore, in the equation $f(x)=0$ we have

$$\Sigma(\alpha^n) = \text{coefficient of } \frac{1}{x^{n+1}} \text{ in } \frac{f'(x)}{f(x)}.$$

Now, by actual division, $\quad \dfrac{3x^2+p}{x^3+px+q} = \dfrac{3}{x}-\dfrac{2p}{x^3}-\dfrac{3q}{x^4}+\dfrac{2p^2}{x^5}+\dfrac{5pq}{x^6}+\&c.;$

therefore $\quad \alpha^2+\beta^2+\gamma^2 = -2p,\quad \alpha^3+\beta^3+\gamma^3 = -3q,$

$$\alpha^4+\beta^4+\gamma^4 = +2p^2,\quad \alpha^5+\beta^5+\gamma^5 = +5pq;$$

$\therefore\quad 3(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5) = -30p^2q = 5(\alpha^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4).$

3069. (Proposed by the EDITOR.)—Find the sides of a triangle in rational numbers, such that the three sides and the area shall be in arithmetical progression.

Solution by Dr. HART.

Let $2x-y$, $2x$, $2x+y$ be the sides of the triangle, then the area is $\left\{3x^2(x+y)(x-y)\right\}^{\frac{1}{2}}$; hence, since the sum of the extremes is equal to the sum of the means, we have

$$2x-y+\left\{3x^2(x+y)(x-y)\right\}^{\frac{1}{2}} = 4x+y,$$

therefore $\quad \left\{3x^2(x+y)(x-y)\right\}^{\frac{1}{2}} = 2(x+y) = \text{area}.$

Squaring and dividing by $x+y$, we have

$$3x^2(x-y) = 4(x+y), \text{ whence } y = \frac{3x^3-4x}{3x^2+4};$$

$$\therefore\quad 2x-y=\frac{3x\,(x^2+4)}{3x^2+4},\ \ 2x,\ \ 2x+y=\frac{x\,(9x^2+4)}{3x^2+4},\ \text{and}\ 2\,(x+y)=\frac{12x^3}{3x^2+4},$$

are general expressions for the sides and area of the triangle, where x may be any number;

if $x=1$, then $y=-\frac{1}{7}$, and the numbers are $\frac{13}{7}$, 2, $\frac{15}{7}$, $\frac{12}{7}$;
if $x=2$, then $y=1$, and the numbers are 3, 4, 5, 6;
if $x=3$, then $y=\frac{69}{31}$, and the numbers are $\frac{117}{31}$, 6, $\frac{255}{31}$, $\frac{324}{31}$.

Supplementary Note on Question 3693. *By* W. S. B. Woolhouse.

On page 25 of this Volume there is a Note by Mr. Carr, intended by him as confirmatory of his method of solving Question 3693, and as destroying the force of the objection I had previously made. That there may be no misunderstanding of the purport of my objection, I have again to repeat the fact, that Mr. Watson's method of solution is undoubtedly quite correct, and that of Mr. Carr as undoubtedly inaccurate. Mr. Carr adduces what he considers to be an illustrative problem respecting black and white balls contained in boxes, which he treats correctly enough, but it is not by any means analogous to Mr. Watson's question, and only further elucidates the incorrectness of Mr. Carr's solution to that Question, if, indeed, any such additional evidence were needed.

4011. (Proposed by J. J. Walker.)—Find the position of the tangent plane to an ellipsoid, such that the tetrahedron formed by it and the three planes of the principal sections has (1) the minimum value; (2) the distance of its centroid from the centre of the ellipsoid a minimum.

I. *Solution by* Professor Townsend, F.R.S.

The three intercepts λ, μ, ν cut off from the centre on the axes of the ellipsoid a, b, c by any tangent plane to the surface being connected by the relation
$$\frac{a^2}{\lambda^2}+\frac{b^2}{\mu^2}+\frac{c^2}{\nu^2}=1,$$
we have, subject to this condition, to determine their values so that $\lambda\mu\nu$ for (1), and $\lambda^2+\mu^2+\nu^2$ for (2), shall be a minimum; hence, by the ordinary process, we get at once, $\lambda^2=3a^2$, $\mu^2=3b^2$, $\nu^2=3c^2$ for the former case, and $\lambda^2=(a+b+c)\,a$, $\mu^2=(a+b+c)\,b$, $\nu^2=(a+b+c)\,c$ for the latter case, and therefore, &c.

II. *Solution by the* Rev. J. L. Kitchin, M.A.

1. The tangent plane $\dfrac{xx'}{a^2}+\dfrac{yy'}{b^2}+\dfrac{zz'}{c^2}=1$ meets the axes x, y, z at

$\frac{a^2}{x'}$, $\frac{b^2}{y'}$, $\frac{c^2}{z'}$; hence $\frac{b^2c^2}{2y'z'}$ = area of face on yz, and therefore $\frac{a^2b^2c^2}{6x'y'z'}$ = solid which is to be a minimum, or $x'y'z'$ = a maximum .... (1),

subject to the condition $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=1$ .... (2).

Differentiating (1) and (2), and dropping accents, we have

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0, \text{ and } \frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0;$$

therefore $\frac{\lambda}{x}=\frac{x}{a^2}$, $\frac{\lambda}{y}=\frac{y}{b^2}$, $\frac{\lambda}{z}=\frac{z}{c^2}$; therefore $3\lambda=1$ and $\lambda=\frac{1}{3}$;

therefore $x=\pm\frac{1}{3}a\sqrt{3}$, $y=\pm\frac{1}{3}b\sqrt{3}$, $z=\pm\frac{1}{3}c\sqrt{3}$.

2. The coordinates of the centre of gravity are $\frac{3a^2}{4x'}$, $\frac{3b^2}{4y'}$, $\frac{3c^2}{4z'}$;

$\frac{a^4}{x^2}+\frac{b^4}{y^2}+\frac{c^4}{z^2}$ = a minimum, with the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Hence $\frac{a^4}{x^3}dx+\frac{b^4}{y^3}dy+\frac{c^4}{z^3}dz=0$, and $\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}+\frac{z\,dz}{c^2}=0$;

$\therefore$ $\lambda\frac{x}{a^2}=\frac{a^4}{x^3}$, &c.; therefore $\lambda=\frac{a^4}{x^2}+\frac{b^4}{y^2}+\frac{c^4}{z^2}=\frac{a^6}{x^4}=\frac{b^6}{y^4}=\frac{c^6}{z^4}$;

therefore $$\frac{a^3}{x^2}=\frac{b^3}{y^2}=\frac{c^3}{z^2}=\frac{a+b+c}{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}=a+b+c;$$

therefore $$x=\frac{a^{\frac{3}{2}}}{(a+b+c)^{\frac{1}{2}}},\quad y=\frac{b^{\frac{3}{2}}}{(a+b+c)^{\frac{1}{2}}},\quad z=\frac{c^{\frac{3}{2}}}{(a+b+c)^{\frac{1}{2}}}.$$

4073. (Proposed by J. C. Malet, M.A.)—If a conic S circumscribes a triangle A, B, C, self conjugate to another conic S′, prove that the vertices of the triangle formed by tangents to S at A, B, C, together with the our points of intersection of S and S′ lie on another conic, and find the equation of this conic, the equations of S and S′ being general.

I. *Solution by* C. Leudesdorf.

Taking $S=2a'yz+2b'zx+2c'xy$, $S'=ax^2+by^2+cz^2$, the coordinates of the triangle formed by the tangents at A, B, C to S are

$$-\frac{x}{a'}=\frac{y}{b'}=\frac{z}{c'},\quad \frac{x}{a'}=-\frac{y}{b'}=\frac{z}{c'},\quad \frac{x}{a'}=\frac{y}{b'}=-\frac{z}{c'} \;\ldots\ldots\; (A', B', C').$$

Hence if A′ lie on the conic $\lambda S+S'=0$, which passes through the points of intersection of S and S′, we have

$$aa'^2+bb'^2+cc'^2-2\lambda a'b'c'=0.$$

The same condition is found for the points B′ and C′. Hence A′, B′, C′ lie on a conic passing through the intersections of S and S′.

II. *Solution by* Professor WOLSTENHOLME, M.A.

Take the equation of S to be $fyz+gzx+hxy=0$, and of S′ to be $ux^2+vy^2+wz^2=0$, then that of any conic through their four common points is $k\text{S}+\text{S}'=0$, and the tangents to S at BC meet in the point $\dfrac{x}{-f}=\dfrac{y}{g}=\dfrac{z}{h}$, at which point $\text{S}=-fgh$, $\text{S}'=uf^2+vg^2+wh^2$, hence the conic $(uf^2+vg^2+wh^2)\,\text{S}+fgh\text{S}'=0$ will pass through the three such points, and the invariants of S, S′ are

$$\Delta=fgh,\quad \Theta=2\,(uf^2+vg^2+wh^2),\quad \Theta'=0,\quad \Delta'=8uvw,$$

hence the general equation required is $\Theta\text{S}+2\Delta\text{S}'=0$, the conics being such that $\Theta'=0$.

3917. (Proposed by Dr. HART.)—Find the least numbers that will make $x^2+xy+y^2=\square$, $x^2+xz+z^2=\square$, and $y^2+yz+z^2=\square$.

I. *Solution by* ARTEMAS MARTIN.

Put $$x^2+xy+y^2=\left(\frac{p}{q}x-y\right)^2,\ \text{then}\ \frac{x}{y}=\frac{q^2+2pq}{p^2-q^2}.$$

Take $$p=2,\ q=1;\ \text{then}\ \frac{x}{y}=\frac{5}{3}.$$

Now let $5m=x$, $3m=y$ and $wm=z$; then the other expressions become, after expunging m^2, $25+5w+w^2=\square$, $9+3w+w^2=\square$ .... (1, 2).

Put (1) $=(nw-5)^2$ and we obtain $w=\dfrac{5\,(2n+1)}{n^2-1}$.

Substituting in (2) and reducing, we have

$$9n^4+30n^3+97n^2+70n+19=\square \ \text{..................... (3).}$$

Putting (3) $=(3n^2+5n+\frac{19}{2})^2$ we get $3n^2-5n=\frac{57}{4}$, which gives $n=\frac{19}{6}$; therefore $w=\frac{264}{65}$, and $z=\dfrac{264m}{65}$.

Now take $m=65$ and we have $x=325$, $y=195$, $z=264$.

II. *Solution by the* PROPOSER.

We have $x^2+xy+y^2=\square$, when $x=m^2-n^2$ and $y=2mn+n^2$,

and $$x^2+xz+z^2=\square,\ \text{when}\ z=\frac{(2pq+q^2)\,x}{p^2-q^2};$$

hence by substitution we obtain

$$y^2+yz+z^2=y^2+\frac{2pq+q^2}{p^2-q^2}xy+\left(\frac{2pq+q^2}{p^2-q^2}\right)^2x^2=\square.$$

Multiplying by $(p^2-q^2)^2$, and arranging the terms according to the powers

of p, we have

$$y^2p^4 + 2xy\,p^3q + (4x^2 + xy - 2y^2)\,p^2q^2 + (4x^2 - 2xy)\,pq^3 + (x^2 - xy + y^2)\,q^4 = \square.$$

Let $m=2$, $n=1$, then $x=3$, $y=5$; whence the above expression becomes

$$25p^4 + 30p^3q + p^2q^2 + 6pq^3 + 19q^4 = \square,$$

which is the case when $p = \frac{1}{2}q$.

Let $p = r + \frac{1}{2}q$, then by substitution and arranging terms we have

$$25r^4 + 80r^3q + \tfrac{167}{2}r^2q^2 + 42rq^3 + \tfrac{441}{16}q^4 = \square = (\tfrac{21}{4}q^2 + 4rq + \tfrac{45}{7}r^2)^2,$$

whence $r = \frac{7}{4}q$, $p = \frac{9}{4}q$, where $p=9$, $q=4$, therefore $z = \frac{264}{65}$; or multiplying by 65, we have $x=195$, $y=325$, $z=264$, which are probably the least numbers.

[Dr. Hart states that by another process he finds 264, 325, 440 as three other values that will satisfy the conditions of the Question.]

III. *Solution by* Asher B. Evans, M.A.

Let $\left(x - \frac{m}{n}y\right)$ be the root of (1); then $\frac{x}{y} = \frac{m^2 - n^2}{2mn + n^2}$.

Assume $m=2$ and $n=1$; then $\frac{x}{y} = \frac{3}{5}$, and by taking $x=3$, $y=5$, the second and third conditions become

$$9 + 3z + z^2 = \square, \quad 25 + 5z + z^2 = \square \quad \ldots\ldots\ldots\ldots\ldots\ldots (4, 5).$$

Let $(pz - 3)$ be the root of (4), and we shall find $z = 3\left(\frac{2p+1}{p^2-1}\right)$; which substituted in (5) gives $25 + 15\left(\frac{2p+1}{p^2-1}\right) + 9\left(\frac{2p+1}{p^2-1}\right)^2 = \square$.

By multiplying by $(p^2-1)^2$ and ordering the terms of the product with reference to p, we have $25p^4 + 30p^3 + p^2 + 6p + 19 = \square$ $\ldots\ldots\ldots\ldots\ldots\ldots (6)$.

Since $p = \frac{1}{2}$ satisfies (6), put $p = q + \frac{1}{2}$ in order to find a value of p greater than unity that will satisfy (6); then

$$25q^4 + 80q^3 + \tfrac{167}{2}q^2 + 42q + \tfrac{441}{16} = \square, \text{ say } = (\tfrac{21}{4} + 4q + \tfrac{45}{7}q^2)^2;$$

from which $q = \frac{7}{4}$, and $p = q + \frac{1}{2} = \frac{9}{4}$, and $z = 3\left(\frac{2p+1}{p^2-1}\right) = \frac{264}{65}$.

It is evident that the values of x, y, z may each be multiplied by 65, or by any other number; therefore $x=195$, $y=325$, and $z=264$.

3970. (Proposed by A. B. Evans, M.A.)—ABC is a plane triangle, and OA, OB, OC are lines making equal angles with one another. Find the least integral values of BC, CA, AB, that will make OA, OB, OC integral.

Solution by Samuel Bills.

Let $OA = x$, $OB = y$, $OC = z$; we shall have

$$AB^2 = x^2 + y^2 - 2xy\cos 120^\circ = x^2 + xy + y^2.$$

Similarly, $\qquad AC^2 = x^2 + xz + z^2$, and $BC^2 = y^2 + yz + z^2$;
so that the question resolves itself into finding
$x^2 + xy + y^2 = \square \ldots (1)$, $\quad x^2 + xz + z^2 = \square \ldots (2)$. $\quad y^2 + yz + z^2 = \square \ldots (3)$;
which is identical with the preceding Question 3917.

Assume $y = \dfrac{p^2-1}{2p+1}x$ and $z = \dfrac{q^2-1}{2q+1}x$; then (1) and (2) will be satisfied, and it will remain to find

$$\left(\frac{p^2-1}{2p+1}\right)^2 + \left(\frac{p^2-1}{2p+1}\right)\left(\frac{q^2-1}{2q+1}\right) + \left(\frac{q^2-1}{2q+1}\right)^2 = \square \ \ldots\ldots\ldots \ (4).$$

Now, whatever q is, (4) will be satisfied by taking $p = \dfrac{1}{q}$; but it will be seen that if q is > 1, so as to make $\dfrac{q^2-1}{2q+1}$ positive, then p will be < 1, and $\dfrac{p^2-1}{2p+1}$, and consequently y would be negative, which must be avoided. For simplicity of working, and as, moreover, we want to find the *least* numbers, take $q = 2$; then (4) will become

$$\left(\frac{p^2-1}{2p+1}\right)^2 + \frac{3}{5}\left(\frac{p^2-1}{2p+1}\right) + \frac{9}{25} = \square;$$

which reduces to $\qquad 25p^4 + 30p^3 + p^2 + 6p + 19 = \square \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (5).$
Since q was taken $= 2$, we know that (5) will be satisfied by $p = \frac{1}{2}$.
Assume, therefore, $p = t + \frac{1}{2}$; substituting this value of p in (5) we get

$$t^4 + \tfrac{16}{5}t^3 + \tfrac{167}{50}t^2 + \tfrac{42}{25}t + \tfrac{441}{400} = \square = \left(\tfrac{9}{7}t^2 + \tfrac{4}{5}t + \tfrac{21}{20}\right)^2 \text{ (suppose).}$$

Reducing the above, we find $t = \frac{7}{4}$, and $p = t + \frac{1}{2} = \frac{9}{4}$.
From this we find $y = \frac{65}{88}x$, and $z = \frac{3}{5}x$; and taking $x = 440$, we shall have $y = 325$ and $z = 264$, which are the *least* numbers I have been able to find. From these numbers we readily find, for the sides of the triangle in Question 3970, $AB = 665$, $AC = 615$, $BC = 511$.

Of course any number of solutions may be found by giving different values to q, though they would be larger than the above.

[Dr. Hart finds $x = 195$, $y = 325$, $z = 264$, see the preceding page (60) of this Volume, and thence $BC = 511$, $CA = 455$, $AB = 399$, which he believes to be the *least* integral values that will satisfy the conditions of the Question.]

4096. (Proposed by the Rev. Dr. Booth, F.R.S.)—Along a line of curvature tangent planes are drawn to a surface of the second order; show that the perpendiculars from the centre on these planes generate a cone of the second order, whose focal lines coincide with the optic axes of the surface, or with the perpendiculars to its circular sections.

I. *Solution by* J. J. Walker, M.A.

Let the line of curvature on the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

be determined by the cone

$$\frac{b^2-c^2}{f}\frac{x^2}{a^2}+\frac{c^2-a^2}{g}\frac{y^2}{b^2}+\frac{a^2-b^2}{h}\frac{z^2}{c^2}=0, \text{ where } f+g+h=0.$$

The locus of the perpendiculars from centre on the tangent planes along this line of curvature is readily found to be

$$\frac{a^2(b^2-c^2)}{f}x^2+\frac{b^2(c^2-a^2)}{g}y^2+\frac{c^2(a^2-b^2)}{h}z^2=0,$$

a cone of the second order, the equations to the focal lines of which are $z=0$ and

$$\frac{x^2}{\dfrac{f}{a^2(b^2-c^2)}-\dfrac{g}{b^2(c^2-a^2)}}+\frac{z^2}{\dfrac{h}{c^2(a^2-b^2)}-\dfrac{g}{b^2(c^2-a^2)}}=0,$$

or $$\frac{a^2(b^2-c^2)x^2}{a^2b^2(f+g)+b^2c^2f+c^2a^2g}+\frac{c^2(a^2-b^2)z^2}{b^2c^2(g+h)-a^2b^2h-c^2a^2g}=0.$$

The relations $f+g=-h$, $g+h=-f$, show that denominators are equal, with opposite signs, or the equations of the focal lines are $z=0$ and

$$a^2(b^2-c^2)x^2-c^2(a^2-b^2)z^2=0,$$

a pair of lines perpendicular to the circular sections of (1).

II. *Solution by* Professor Wolstenholme, M.A.

The equation of a tangent plane is $\dfrac{xX}{a^2}+\dfrac{yY}{b^2}+\dfrac{zZ}{c^2}=1$,

where (X, Y, Z) satisfy the equations

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}=1,\quad \frac{X^2}{a^2+\lambda}+\frac{Y^2}{b^2+\lambda}+\frac{Z^2}{c^2+\lambda}=1;$$

and the equation of the central perpendicular will be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

l, m, n satisfying the equations

$$l^2a^2+m^2b^2+n^2c^2=1,\quad \frac{l^2a^4}{a^2+\lambda}+\frac{m^2b^4}{b^2+\lambda}+\frac{n^2c^4}{c^2+\lambda}=1,$$

whence $$\frac{l^2a^2}{a^2+\lambda}+\frac{m^2b^2}{b^2+\lambda}+\frac{n^2c^2}{c^2+\lambda}=0,$$

and the equation of the cone generated is

$$\frac{a^2x^2}{a^2+\lambda}+\frac{b^2y^2}{y^2+\lambda}+\frac{c^2z^2}{c^2+\lambda}=0,$$

whose focal lines are perpendicular to the circular sections.

4148. (Proposed by W. H. H. Hudson, M.A.)—The perpendiculars from the angles A, B, C meet the sides of a triangle in P, Q, R: prove that

the centre of gravity of six particles proportional respectively to $\sin^2 A$, $\sin^2 B$, $\sin^2 C$, $\cos^2 A$, $\cos^2 B$, $\cos^2 C$, placed at A, B, C, P, Q, R, coincides with that of the triangle PQR.

I. *Solution by* R. TUCKER, M.A.

Refer the triangle to AP, PC as axes; then

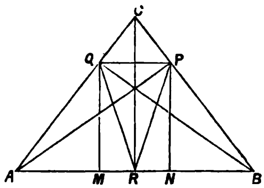
$$3\bar{x} = b\sin^2 C\cos C - c\sin^2 B\cos B + (c\cos C\cos^2 B - b\cos B\cos^2 C)\cos A$$
$$= 2R\cos^2 A\sin(C \backsim B),$$
$$3\bar{y} = \sin A\cos B\cos C\,(c\cos B + b\cos C) + b\sin^2 A\sin C$$
$$= a\sin A(\cos B\cos C + \sin B\sin C) = a\sin A\cos(B \backsim C);$$

but the centre of gravity of the triangle PQR is the same as that of three equal particles at P, Q, R; therefore

$$3\bar{x}_1 = (c\cos C - b\cos B)\cos A = 2R\cos^2 A\sin(C \backsim B),$$
$$3\bar{y}_1 = (c\cos C + b\cos B)\sin A = a\sin A\cos(B \backsim C);$$

hence the truth of the theorem.

II. *Solution by the* Rev. J. L. KITCHIN, M.A.



Take A as origin and AB as axis of x.

Then the x of A $= 0$,

„ R $= b\cos A$,

„ P $= c\sin^2 B$,

„ B $= c$,

„ C $= b\cos A$,

„ Q $= c\cos^2 A$;

$\therefore\ \bar{x} = \frac{1}{3}(c\sin^2 B + b\cos A + c\cos^2 A)$,

which is also the abscissa of the centre of gravity of the triangle PQR.

Similarly for $\bar{y}$.

2981. (Proposed by the Rev. G. H. HOPKINS, M.A.)—If the sum of the squares of two consecutive integers be equal to the square of another integer, find their general values, and show how to find any number of particular solutions.

Solution by Dr. HART.

Let x and $x+1$ be the sides of two squares, then the side of a third square, equal to the sum of the two squares, is $(2x^2+2x+1)^{\frac{1}{2}}$, and we have

to make $$2x^2+2x+1 = \square = \left(\frac{p}{q}x - 1\right)^2, \text{ suppose.}$$

Reducing, we find $$x = \frac{2pq+2q^2}{p-2q^2},$$

therefore $\quad x+1=\dfrac{p^2+2pq}{p^2-2q^2}$, and $(2x^2+2x+1)^{\frac{1}{2}}=\dfrac{p^2+2pq+2q^2}{p^2-2q^2}$,

which are general expressions for the roots of the numbers. But in order to have integers, let $p^2-2q^2=\pm 1$.

Having found the two first values of p and q in equations of this kind, every succeeding set may be found by the following formulae:—

For $p^2-Nq^2=\pm 1$ the values are $2mr+r'$, and $2ms+s'$; and for $p^2-Nq^2=1$ the succeeding values are $2mr+r'$, and $2ms-s'$; m being always the value of p in the second set, r and $s=$ the last found values of p and q, and r' and $s'=$ the next preceding values.

In $p^2-2q^2=\pm 1$, the first values of p, q are 1, 0, the second values are 1, 1 (m being here $=1$), and the succeeding values are $\begin{matrix}3 & 7 & 17 & 41 & 99 & 239\\ 2 & 5 & 12 & 29 & 70 & 169\end{matrix}$ &c., *ad infinitum*, all found by the first formulæ, and these values of p and q being substituted in the numerators of x, $x+1$, and $(2x^2+2x+1)^{\frac{1}{2}}$, will give integral answers to the problem.

In $p^2-3q^2=1$, the negative sign has no place, because the second values of p and q answer the positive sign, and all succeeding values answer the same sign, for reasons well understood. When the second values answer the negative sign, the values of p and q will answer the positive and negative signs alternately. The first values of p and q in $p^2-3q^2=1$ are $p=1$, $q=0$, the second are $p=2$, $q=1$. Here $m=2$, and the succeeding values, found by the second formulæ, are $\begin{matrix}7 & 26 & 97 & 362 & 1351\\ 4 & 15 & 56 & 209 & 985\end{matrix}$ &c., *ad infinitum;* and so we might proceed for any value of N, not a square. In $p^2-940751q^2=1$, the first values of p, q are $p=1$, $q=0$, but the second values are numbers of immense magnitude, and were found thirty years ago by a student of Prof. C. Gill, of St. Paul's College, at Flushing, near New York. These numbers are said to be the largest of the kind that have ever been found. Mr. A. B. Evans, M.A., has also accomplished the same task, and his result has been published in one of the Volumes of the *Reprint* from the *Educational Times.*

In this way we can find the successive values of p, q more expeditiously than by any other method; and then we obtain integral values for the roots of the required squares.

In the general equation $p^2-Nq^2=\pm 1$, the first values of p and q are $p=1$, $q=0$; the second values are found, either by trial, when the value of N is small, or by developing $\sqrt{N}$ into a continued fraction, finding the partial quotients, and from these forming a series of converging fractions, &c. Having found these two sets of values of p, q, all other sets are found successively by the formulæ above mentioned.

The advantage of this method of finding p and q (after the first two sets have been ascertained) over other methods, appears conspicuously when N is a large number.

[Dr. Hart remarks that he has not demonstrated the correctness of the formulæ above given, but that he has tested their accuracy by submitting them to a very great number of trials, and has found them to hold good in every case.]

3951. (Proposed by Dr. Hart.)—Find a quadrilateral inscribable in a circle, whose sides and diagonals shall be integers.

I. *Solution by* Asher B. Evans, M.A.

Let ABCD be the quadrilateral; then, since it is inscribable in a circle, we have from the geometry of the figure,

$$AC^2 = (AD \cdot BC + AB \cdot DC)\left(\frac{AD \cdot AB + BC \cdot CD}{AD \cdot BC + AD \cdot DC}\right) \quad \ldots\ldots\ldots (1),$$

$$BD^2 = (AD \cdot BC + AB \cdot DC)\left(\frac{AB \cdot BC + AD \cdot DC}{AD \cdot AB + BC \cdot CD}\right) \quad \ldots\ldots\ldots (2).$$

Put $AB = x$, $BC = mx$, $CD = nx$, $AD = px$; then (1) and (2) become

$$AC^2 = (mp+n)\left(\frac{p+mn}{m+np}\right)x^2 \text{ and } BD^2 = (mp+n)\left(\frac{m+np}{p+mn}\right)x^2.$$

Let $\dfrac{p+mn}{m+np} = a^2$, then $p = \left(\dfrac{n-a^2}{a^2n-1}\right)m$; and AC and BD will be rational, if $mp+n = \left(\dfrac{n-a^2}{a^2n-1}\right)m^2+n = \square$. Put $n=q^2$, and let

$$\left(\frac{q^2-a^2}{a^2q^2-1}\right)m^2+q^2 = \left(\frac{my}{a^2q^2-1}+q\right)^2; \text{ then } m = \frac{2qy\,(a^2q^2-1)}{(q^2-a^2)(a^2q^2-1)-y^2}.$$

For a particular solution take $a = \frac{3}{2}$, $y = \frac{3}{2}$, $q = 2$, then $m = \frac{192}{47}$, $n = 4$, $p = \frac{42}{47}$, $AC = \frac{585}{141}x$, $BD = \frac{260}{141}x$. Let $x = 141$; then $AB = 141$, $BC = 576$, $CD = 564$, $AD = 126$, $AC = 585$, and $BD = 260$.

II. *Solution by the* Proposer.

Let a, b, c, d be the sides, and x, y the diagonals; then, by well known properties, we have $xy = ac+bd$, and $x : y = bc+ad : ab+cd$;

whence $$x = \frac{ac+bd}{y} = \frac{(bc+ad)\,y}{ab+cd},$$

and $$y^2 = \frac{(ab+cd)(ac+bd)}{bc+ad} = \frac{(ab+cd)(ac+bd)(bc+ad)}{(bc+ad)^2};$$

therefore $(ab+cd)(ac+bd)(bc+ad)$ must be a square number, or, multiplying and arranging terms, we have

$$a^2b^2c^2+abc\,(a^2+b^2+c^2)\,d+(a^2b^2+a^2c^2+b^2c^2)\,d^2+abcd^3 = \square,$$

put it $= \left(abc+\dfrac{a^2+b^2+c^2}{2}d\right)^2$, whence we find $d = \dfrac{(a^2-b^2+c^2)^2-4a^2c^2}{4abc}$.

Let $a, b, c = 2, 3, 7$ respectively, then $d = \frac{48}{7}$, $y = \frac{22}{3}$, $x = \frac{33}{7}$; or, multiplying by 21, we have in integers 42, 63, 147, 144 for the sides, and 99, 154 for the diagonals.

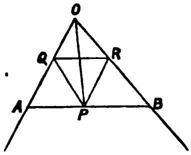
Let $a, b, c = 1, 4, 7$, then $d = \frac{60}{7}$, $y = \frac{17}{2}$, $x = \frac{34}{7}$; or, multiplying by 14, we have 14, 56, 98, 120 for the sides, and 68, 119 for the diagonals.

In the suppositions for a, b, c, care must be taken that they be such that d shall be less than $a+b+c$; or, in general, that the sum of any three of the quantities a, b, c, d shall be greater than the fourth.

4182. (Proposed by the Rev. A. F. Torry, M.A.)—P is a point within the acute angle AOB, and a ray of light PQR emanating from P is re-

flected at OA, OB, in succession, and returns to P; show that the length of its path is 2OP sin AOB, and that OP bisects the angle QPR.

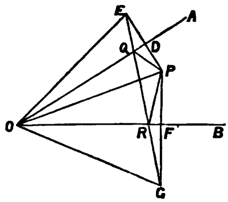
I. *Solution by* H. S. MONCK; A. RENSHAW; H. I. B.; *and others.*



Draw APB bisecting the external angles at P; then the triangle AOB is evidently that formed by the external bisectors of the angles of the triangle PQR, and the line OP joining P to O, the intersection of a pair of external bisectors, is known to be the internal bisector of the third angle QPR.

Again, the line PO joining the vertex P to the intersection of external bisectors is known to be equal to $s \sec \frac{1}{2}$PQR, where s is half the sum of the sides of the triangle PQR; and the angle AOB $= 90° - \frac{1}{2}$PQR, or $\frac{1}{2}$PQR $= 90° -$ AOB; therefore $s =$ PO sec $\frac{1}{2}$PQR = PO sin AOB; and the sum of the sides of the triangle PQR is $2s$; therefore the whole distance traversed by the ray is 2OP sin AOB; which proves the theorem.

II. *Solution by* J. L. MCKENZIE.



Draw PDE, PFG perpendicular to OA, OB, and double of PD, PF respectively. Draw EQRG; then PQRP is evidently the path of the ray. The line OE = OP = OG; therefore the angle OPQ = OEQ = OGR = OPR.

The whole path = EG = 2OE sin $\frac{1}{2}$EOG
= 2OP sin AOB.

2031. (Proposed by the EDITOR.)—Show (1) the least positive value of x that will make $927x^2 - 1236x + 413$ a rational square is $\frac{22421}{3}$, and (2) that no integral values of x can be found to satisfy the condition.

I. *Solution by* DR. HART.

1. From the given expression subtract $(30x-19)^2$, and the remainder is $27x^2 - 96x + 52$, and since $(96)^2 - 4 \cdot 27 \cdot 52 = 3600 = \square$, the expression can be divided into the factors $3x-2$ and $9x-26$. Thus we have

$$927x^2 - 1236x + 413 = (30x-19)^2 + (3x-2)(9x-26) = \square$$
$$= \left\{ (30x-19) - \frac{m}{n}(9x-26) \right\}^2, \text{ suppose;}$$

whence, reducing we find $\quad x = \dfrac{26m^2 - 38mn - 2n^2}{9m^2 - 60mn - 3n^2}$.

Put $9m^2 - 60mn - 3n^2 = 1$; then, transposing and dividing by 9, we have

$$m^2 - \tfrac{20}{3}nm = \tfrac{1}{9}(3n^2+1), \text{ whence } m = \tfrac{1}{3}\left\{10n \pm (103n^2+1)\right\}.$$

To have rational values, put $103n^2+1=p^2$, then $p^2-103n^2=1$. Develope $\sqrt{103}$ into a continued fraction, and the partial quotients will be 10, 6, 1, 2, 1, 1, 9, 1, 1, 2, 1, 6, 20, from which form a series of converging fractions, and that fraction standing under 20 in an even place is $\frac{227528}{22419}$; therefore $p=227528$, $n=22419$, hence $m=\frac{10n\pm p}{3}=\frac{451718}{3}$, or $-\frac{3338}{3}$. Using the negative value of m, $x=-\frac{226162846}{9}$, which is its least value. Using the positive value of m, we find $x=\frac{4243018104458}{9}$.

2. Now, since n is divisible by 3, and p is *not*, m must be a fraction, and also x. And in the general formulæ (p. 383 of Barlow's *Theory of Numbers*), viz., $\frac{(p+qN^{\frac{1}{2}})^m+(p-qN^{\frac{1}{2}})^m}{2}$, and $\frac{(p+qN^{\frac{1}{2}})^m-(p-qN^{\frac{1}{2}})^m}{2N^{\frac{1}{2}}}$, the first term not being divisible by 3, and all the other terms divisible by 3, therefore no integral value of x can be found to satisfy the condition.

[Dr. Hart remarks that "the foregoing method of solving such problems is in part new. Putting the denominator in the value of x equal to unity has never, so far as I know, been done before. Such problems have usually been solved by Gauss's method of Congruous Numbers, as laid down in his *Disquisitiones Arithmeticæ*,—a method which is very abstruse, and quite beyond the reach of elementary mathematicians."]

II. *Solution by the* Proposer.

1. Putting the given expression in the form

$$103\,(3x-2)^2+1=\square=p^2,\ \text{or}\ 103q^2+1=p^2,$$

we have to find the values of p and q that will satisfy the condition

$$p^2-103q^2=1.$$

By the usual method, see Barlow as cited by Dr. Hart, the least values of p and q are found to be

$$p'=227528,\quad q'=22419;$$

thus the least positive value of x that will make the given expression a rational square is $$x=\tfrac{1}{3}(q'+2)=\frac{22421}{3}.$$

2. The general value of q is

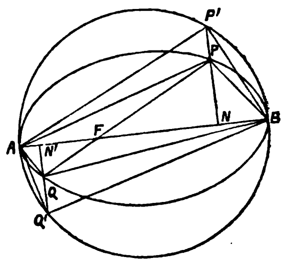
$$q=\frac{\{p'+q'(103)^{\frac{1}{2}}\}^m-\{p'-q'(103)^{\frac{1}{2}}\}^m}{2\,(103)^{\frac{1}{2}}},$$

where m is any whole number.

And as q' is a multiple of 3, it follows that q, which is a multiple of q', is also a multiple of 3, and therefore no value of q can make x, or $\frac{1}{3}(q+2)$, integral.

4166. (Proposed by the Editor.)—If AB be the major axis of an ellipse, and PFQ a focal chord, prove that FP tan APB = FQ tan AQB.

Solution by R. W. GENESE, B.A.



Let the angle PAB $= \theta$ and PBA $= \phi$. If the direction of AP be fixed, the direction of BP is known, and *vice versâ;* therefore there must be a linear relation between $\tan\theta$ and $\tan\phi$ of the form

$$A\tan\theta\tan\phi + B\tan\theta + C\tan\phi + D = 0 \quad\dots\dots\dots (1),$$

where A, B, C, D are constants.

But when $\theta = 0$, $\phi = \frac{1}{2}\pi$, therefore $C = 0$, and similarly $B = 0$; hence, from (1), we obtain $\tan\theta\tan\phi = \text{constant}\dots\dots (2)$.

Put P on the minor axis and we see that this constant $= \dfrac{b^2}{a^2}$.

Draw PN perpendicular to AB; let PN $= y$, AN $= x$; NB $= x'$; then the relation (2) becomes

$$\frac{y}{x}\cdot\frac{y}{x'} = \frac{b^2}{a^2}\quad\dots\dots\dots (3),$$

which is a well known proposition in Geometrical Conics.

Now $$\tan APB = -\tan(\theta+\phi) = -\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi},$$

$$\propto \frac{y}{x}+\frac{y}{x'} \quad \propto y\left(\frac{x+x'}{xx'}\right);$$

therefore $y\tan APB = \text{constant}$, whence the theorem follows at once.

[Other proofs of this theorem are given on pp. 40, 41 of this volume. We have drawn the figure to the foregoing solution so as to illustrate Mr. TUCKER's solution also.]

3663. (Proposed by R. W. GENESE, B.A.)—BaC, CbA, AcB are similar triangles drawn on the sides of a triangle ABC; show that Aa, Bb, Cc meet in the same point.

Solution by Professor TOWNSEND, M.A., F.R.S.

When the three similar triangles BaC, CbA, AcB on the three sides BC, CA, AB of the original triangle ABC are all *isosceles*, this property may be easily proved as follows. Since, evidently,

$$\sin BAa : \sin CAa = \sin ABa : \sin ACa,$$
$$\sin CBb : \sin ABb = \sin BCb : \sin BAb,$$
$$\sin ACc : \sin BCc = \sin CAc : \sin CBc;$$

therefore, from the evident equality of the three pairs of angles BAb and CAc, CBc and ABa, ACa and BCb, consequent on the similarity supposed,

$$\frac{\sin BAa}{\sin CAa}\cdot\frac{\sin CBb}{\sin ABb}\cdot\frac{\sin ACc}{\sin BCc} = 1,$$

and therefore, &c.

[In the form here treated by Mr. TOWNSEND, the theorem is identical with the EDITOR's Question 1281, of which solutions are given in the *Reprint*, Vol. XVIII., p. 111, and Vol. XIX., p. 97, where it is further shown that for varying isosceles triangles, the locus of the point of intersection is a rectangular hyperbola circumscribed about the triangle ABC.]

3981. (Proposed by Professor WOLSTENHOLME.)—A bag contains m white balls and n black balls, and balls are to be drawn from it so long as all drawn are of the same colour. If this be white, A pays B x shillings for the first, rx or the second, $\frac{r(r+1)}{2}x$ for the third, $\frac{r(r+1)(r+2)}{\lfloor 3}x$ for the fourth, and so on; but if black, B pays A y shillings for the first, ry for the second, and so on. Prove that the value of A's expectation at the beginning of the drawing is, in shillings,

$$\frac{\lfloor m+n+r-1 \ \lfloor m \ \lfloor n}{\lfloor m+n \ \lfloor m+r \ \lfloor n+r}\left\{n(n+1)\ldots(n+r)\,y-m(m+1)\ldots(m+r)\,x\right\}.$$

Solution by the PROPOSER.

A's expectation is

$$y\left(\frac{n}{m+n}+r.\frac{n(n-1)}{(m+n)(m+n-1)}+\frac{r(r+1)}{2}\,\frac{n(n-1)(n-2)}{(m+n)(m+n-1)(m+n-2)}+\ldots\ldots \text{ to } n \text{ terms}\right),$$

$$-x\left(\frac{m}{m+n}+r\,\frac{m(m-1)}{(m+n)(m+n-1)}+\frac{r(r+1)}{\lfloor 2}\,\frac{m(m-1)(m-2)}{(m+n)(m+n-1)(m+n-2)}+\ldots\ldots \text{ to } m \text{ terms}\right).$$

Now to sum the series here involved, consider another case in which, with m white and n black balls to draw from, A draws continually without replacing until he draws a white ball, when he is to receive x. His whole expectation is clearly x, and if we subdivide this into the separate expectations from the first, second, &c. ... $(n+1)$th, we have

$$x=x\left(\frac{m}{m+n}+\frac{n}{m+n}\cdot\frac{m}{m+n-1}+\frac{n(n-1)}{(m+n)(m+n-1)}\cdot\frac{m}{m+n-2}+\ldots\ldots \text{ to } n+1 \text{ terms}\right),$$

$$\therefore\ \frac{m+n}{m}=1+\frac{n}{m+n-1}+\frac{n(n-1)}{(m+n-1)(m+n-2)}+\ldots\ldots \text{ to } n+1 \text{ terms},$$

or writing $m+1$ for m,

$$\frac{m+n+1}{m+1}=1+\frac{n}{m+n}+\frac{n(n-1)}{(m+n)(m+n-1)}+\ldots\ldots \text{ to } n+1 \text{ terms}.$$

Hence $$\frac{n}{m+n}\cdot\frac{m+n}{m+1}=\frac{n}{m+n}+\frac{n(n-1)}{(m+n)(m+n-1)}+\ldots\ldots \text{ to } n \text{ terms},$$

$$\frac{n(n-1)}{(m+n)(m+n-1)} \cdot \frac{m+n-1}{m+1} = \frac{n(n-1)}{(m+n)(m+n-1)} + \ldots\ldots \text{ to } n-1 \text{ terms,}$$

&c. = ...

hence, by addition, we have

$$\frac{m+n+1}{m+1}\left\{1 + \frac{n}{m+n+1} + \frac{n(n-1)}{(m+n+1)(m+n)} + \ldots\ldots \text{ to } n+1 \text{ terms}\right\}$$

$$= 1 + 2\frac{n}{m+n} + 3\frac{n(n-1)}{(m+n)(m+n-1)} + 4\frac{n(n-1)(n-2)}{(m+n)(m+n-1)(m+n-2)} + \ldots\ldots \text{ to } n+1 \text{ terms}$$

$$= \frac{m+n+1}{m+1} \cdot \frac{m+n+2}{m+2}, \text{ by the identity proved above.}$$

Proceeding in exactly the same manner, we get

$$1 + 3\frac{n}{m+n} + 6\frac{n(n-1)}{(m+n)(m+n-1)} + 10\frac{n(n-1)(n-2)}{(m+n)(m+n-1)(m+n-2)} + \ldots$$

$$= \frac{(m+n+1)(m+n+2)(m+n+3)}{(m+1)(m+2)(m+3)},$$

and so on, or generally

$$1 + r\frac{n}{m+n} + \frac{r(r+1)}{\lfloor 2} \cdot \frac{n(n-1)}{(m+n)(m+n-1)} + \ldots\ldots \text{ to } n+1 \text{ terms}$$

$$= \frac{(m+n+1)(m+n+2)\ldots(m+n+r)}{(m+1)(m+2)\ldots(m+r)};$$

whence A's expectation is

$$\frac{ny}{m+n} \cdot \frac{(m+n)(m+n+1)\ldots(m+n+r-1)}{(m+1)\ldots(m+r)} - \frac{mx}{m+n} \cdot \frac{(m+n)\ldots(m+n+r-1)}{(n+1)\ldots(n+r)},$$

$$\text{or } \frac{\lfloor m+n+r-1}{\lfloor m+n} \cdot \frac{\lfloor m \; \lfloor n}{\lfloor m+r \; \lfloor n+r}\left\{n(n+1)\ldots(n+r)y - m(m+1)\ldots(m+r)x\right\}.$$

3123. (Proposed by ARTEMAS MARTIN.)—Find the mean distance of the vertex of a right pyramid whose base is a rectangle (1) from all the points in its base, and (2) from all the points within the pyramid, and show that the second mean is three-fourths of the first.

Solution by the PROPOSER.

1. Let a be the altitude of the pyramid, $2b$ and $2c$ the breadth and length of its base, x and y coordinates of any point P_1 in the base, and M_1, M_2

the mean distances required. The distance of P_1 from the vertex of the pyramid is $(a^2+x^2+y^2)^{\frac{1}{2}}$, and

$$M_1 = \frac{1}{bc}\int_0^c\int_0^b (a^2+x^2+y^2)^{\frac{1}{2}}\,dy\,dx$$

$$= \frac{1}{2bc}\int_0^c \left[b\,(a^2+b^2+y^2)^{\frac{1}{2}} + (a^2+y^2)\log\left(\frac{b+(a^2+b^2+y^2)^{\frac{1}{2}}}{(a^2+y^2)^{\frac{1}{2}}}\right)\right] dy$$

$$= \tfrac{1}{4}\,(a^2+b^2+c^2)^{\frac{1}{2}} + \frac{(a^2+b^2)}{4c}\log\left(\frac{c+(a^2+b^2+c^2)^{\frac{1}{2}}}{(a^2+b^2)^{\frac{1}{2}}}\right)$$

$$+ \frac{1}{2bc}\int_0^c (a^2+y^2)\log\left(\frac{b+(a^2+b^2+y^2)^{\frac{1}{2}}}{(a^2+y^2)^{\frac{1}{2}}}\right) dy.$$

$$\int (a^2+y^2)\log\left(\frac{b+(a^2+b^2+y^2)^{\frac{1}{2}}}{(a^2+y^2)^{\frac{1}{2}}}\right) dy = y\,(a^2+\tfrac{1}{3}y^2)\log\left(\frac{b+(a^2+b^2+y^2)^{\frac{1}{2}}}{(a^2+y^2)^{\frac{1}{2}}}\right)$$

$$+\tfrac{1}{3}b\int \frac{y^2\,(3a^2+y^2)\,dy}{(a^2+y^2)\,(a^2+b^2+y^2)^{\frac{1}{2}}}.$$

$$\int \frac{y^2\,(3a^2+y^2)\,dy}{(a^2+y^2)\,(a^2+b^2+y^2)^{\frac{1}{2}}} = \int \frac{y^2dy}{(a^2+b^2+y^2)^{\frac{1}{2}}} + 2a^2\int\frac{dy}{(a^2+b^2+y^2)^{\frac{1}{2}}}$$

$$-\,2a^4\int\frac{dy}{(a^2+y^2)(a^2+b^2+y^2)^{\frac{1}{2}}},$$

all of which, except the last, are simple forms.

Let $\dfrac{aw}{(a^2+b^2+w^2)^{\frac{1}{2}}} = y$, then $w = \dfrac{(a^2+b^2)^{\frac{1}{2}}\,y}{(a^2+y^2)^{\frac{1}{2}}}$, $dy = \dfrac{a\,(a^2+b^2)\,dw}{(a^2+b^2-w^2)^{\frac{3}{2}}}$,

and by substitution and reduction

$$\int\frac{dy}{(a^2+y^2)\,(a^2+b^2+y^2)^{\frac{1}{2}}} = \frac{1}{a}\int\frac{dw}{[(a^2+b^2)^2-b^2w^2]^{\frac{1}{2}}} = \frac{1}{ab}\sin^{-1}\left(\frac{bw}{a^2+b^2}\right)$$

$$= \frac{1}{ab}\sin^{-1}\left(\frac{by}{(a^2+b^2)^{\frac{1}{2}}\,(a^2+y^2)^{\frac{1}{2}}}\right);$$

therefore $$M_1 = \tfrac{1}{3}\,(a^2+b^2+c^2)^{\frac{1}{2}} + \left(\frac{3a^2+c^2}{6b}\right)\log\left(\frac{b+(a^2+b^2+c^2)^{\frac{1}{2}}}{(a^2+c^2)^{\frac{1}{2}}}\right)$$

$$+\left(\frac{3a^2+b^2}{6c}\right)\log\left(\frac{c+(a^2+b^2+c^2)^{\frac{1}{2}}}{(a^2+b^2)^{\frac{1}{2}}}\right) - \frac{a^3}{3bc}\sin^{-1}\left(\frac{bc}{(a^2+b^2)^{\frac{1}{2}}\,(a^2+c^2)^{\frac{1}{2}}}\right).$$

2. Let x, y, z be coordinates of any point P_2 within the pyramid, the vertex being the origin; then $(x^2+y^2+z^2)^{\frac{1}{2}}$ is the distance of P_2 from the vertex, and

$$M_2 = \frac{1}{\frac{1}{3}abc}\int_0^a\int_0^{\frac{cz}{a}}\int_0^{\frac{bz}{a}} (x^2+y^2+z^2)^{\frac{1}{2}}\, dz\, dy\, dx$$

$$= \frac{3}{2abc}\int_0^a\int_0^{\frac{cz}{a}} \left[\frac{bz}{a}\left(\frac{b^2z^2}{a^2}+z^2+y^2\right)^{\frac{1}{2}}\right.$$

$$\left. + (y^2+z^2)\log\left\{\frac{\frac{bz}{a}+\left(\frac{b^2z^2}{a^2}+z^2+y^2\right)^{\frac{1}{2}}}{(y^2+z^2)^{\frac{1}{2}}}\right\}\right] dz\, dy$$

$$= \tfrac{3}{16}(a^2+b^2+c^2)^{\frac{1}{2}} + \tfrac{3}{16}\left(\frac{a^2+b^2}{c}\right)\log\left(\frac{c+(a^2+b^2+c^2)^{\frac{1}{2}}}{(a^2+b^2)^{\frac{1}{2}}}\right)$$

$$+ \frac{3}{2abc}\int_0^a\int_0^{\frac{cz}{a}} (y^2+z^2)\log\left\{\frac{\frac{bz}{a}+\left(\frac{b^2z^2}{a^2}+z^2+y^2\right)^{\frac{1}{2}}}{(y^2+z^2)^{\frac{1}{2}}}\right\} dz\, dy.$$

$$\int_0^a\int_0^{\frac{cz}{a}} (y^2+z^2)\log\left\{\frac{\frac{bz}{a}+\left(\frac{b^2z^2}{a^2}+z^2+y^2\right)^{\frac{1}{2}}}{(y^2+z^2)^{\frac{1}{2}}}\right\} dz\, dy$$

$$= \int_0^a \left[\frac{cz^3}{3a^3}(3a^2+c^2)\log\left(\frac{b+(a^2+b^2+c^2)^{\frac{1}{2}}}{(a^2+c^2)^{\frac{1}{2}}}\right) + \frac{bcz^3}{6a^3}(a^2+b^2+c^2)^{\frac{1}{2}}\right.$$

$$\left. + \frac{b(3a^2-b^2)z^3}{6a^3}\log\left(\frac{c+(a^2+b^2+c^2)^{\frac{1}{2}}}{(a^2+b^2)^{\frac{1}{2}}}\right) - \frac{2z^3}{3}\sin^{-1}\left(\frac{bc}{(a^2+b^2)^{\frac{1}{2}}(a^2+c^2)^{\frac{1}{2}}}\right)\right] dz;$$

therefore $M_2 = \frac{1}{4}(a^2+b^2+c^2)^{\frac{1}{2}} + \left(\frac{3a^2+c^2}{8b}\right)\log\left(\frac{b+(a^2+b^2+c^2)^{\frac{1}{2}}}{(a^2+c^2)^{\frac{1}{2}}}\right)$

$$+ \left(\frac{3a^2+b^2}{8c}\right)\log\left(\frac{c+(a^2+b^2+c^2)^{\frac{1}{2}}}{(a^2+b^2)^{\frac{1}{2}}}\right) - \frac{a^3}{4bc}\sin^{-1}\left(\frac{bc}{(a^2+b^2)^{\frac{1}{2}}(a^2+c^2)^{\frac{1}{2}}}\right).$$

Hence we have $M_2 = \frac{3}{4}M_1$.

[Mr. MARTIN's result in (2) may be readily obtained from that in (1) by the following simple method:—

The number of points in a slice of thickness dz, parallel to the base of the pyramid and at a distance z from the vertex, is

$$\left(\frac{z}{a}\cdot 2b\right)\left(\frac{z}{a}\cdot 2c\right) dz, \text{ or } \frac{4bc}{a^2}z^2dz;$$

and since the distance of each of these points from the vertex is to that of the corresponding point in the base of the pyramid as $z : a$, it follows that the sum of the distances of all the points in the slice from the vertex is

$$\frac{4bc}{a^2}\cdot\frac{z}{a}\cdot M_1\cdot z^2dz, \text{ or } \frac{4bcM_1}{a^3}z^3dz;$$

therefore $$M_2 = \int_0^a \frac{4bcM_1}{a^3}z^3dz \div \tfrac{1}{3}abc = \tfrac{3}{4}M_1.$$

In Mr. WATSON's Solution of this Question, given on p. 40 of Vol. XVIII. of the *Reprint*, the result is incorrectly given as $M_2 = \frac{1}{4}M$.]

4162. (Proposed by Colonel JOHN H. FRY.) — Find x, y, z from the equations $\quad x+y+z=a, \quad x^2+y^2+z^2=b^2, \quad x^3+y^3+z^3=c^3.$

I. *Solution by* T. T. WILKINSON, F.R.A.S.

The given equations may be put under the forms

$$x+y=a-z, \quad x^2+y^2=b^2-z^2, \quad x^3+y^3=c^3-z^3 \ldots\ldots (1, 2, 3).$$

$$(1)\times(2) \text{ gives } x^3+x^2y+xy^2+y^3=ab^2-b^2z-az^2+z^3 \ldots\ldots\ldots (4).$$

$$(1)^3 \text{ gives } x^3+3x^2y+3xy^2+y^3=a^3-3a^2z+3az^2-z^3 \ldots\ldots\ldots (5).$$

$$3(4)-(5) \text{ gives } 2x^3+2y^3=3ab^2-a^3+3a^2z-3b^2z-6az^2+4z^3 \ldots (6).$$

$$(6)=2(3) \text{ gives } 6z^3-6az^2+(3a^2-3b^2)z=a^3-3ab^2+2c^3 \ldots\ldots (7).$$

Hence $\qquad z^3-az^2+\frac{1}{2}(a^2-b^2)z=\frac{1}{6}(a^3-3ab^2+2c^3) \ldots\ldots\ldots\ldots\ldots (8).$

The three roots of (8) give the values of x, y, z.

II. *Solution by the* Rev. Dr. BOOTH, F.R.S.

In the given equations, subtract (2) from the square of (1), multiply the result by z, and substitute for $x+y$ its value $a-z$, then we have

$$2xyz+2(x+y)z^2=(a^2-b^2)z, \quad \text{or} \quad 2xyz=2z^3-2az^2+(a^2-b^2)z \ldots (9).$$

Multiply (2) by (1), and subtract (3) from the product; then

$$xy(x+y)+yz(y+z)+xz(x+z)=ab^2-c^3,$$

or $\qquad a(xy+yz+xz)=3ayz+ab^2-c^3;$

but $\qquad xy+yz+xz=\frac{1}{2}(a^2-b^2);$

therefore $\;a(a^2-b^2)=6xyz+2ab^2-2c^3, \;$ or $\; 6xyz=a^3-3ab^2+2c^3 \ldots\ldots (10).$

Eliminating xyz from (9) and (10), we shall have the final resulting cubic equation in z, which is

$$z^3-az^2+\tfrac{1}{2}(a^2-b^2)z-\tfrac{1}{6}(a^3-3ab^2+2c^3)=0 \ldots\ldots\ldots\ldots\ldots (11).$$

If we assume x, y, z as the three roots of a cubic equation, then, as the sum of the roots with their signs changed is equal to a, the product of the roots taken two by two is $\frac{1}{2}(a^2-b^2)$, and the product of the roots with their signs changed is $\frac{1}{6}(a^3-3ab^2+2c^3)$, these quantities must be the coefficients of the cubic equation which determines these roots; hence the cubic equation which gives the values of x, y, z must be the equation (11) above.

When the sum of the roots is equal to 0, or the second term vanishes, then $a=0$, and the preceding equation becomes

$$z^3-\tfrac{1}{2}b^2\, z-\tfrac{1}{3}c^3=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

Comparing this with the normal form of the cubic equation $z^3-pz-q=0$, it will follow that p represents half the sum of the squares of the three roots, and q represents the third of the sum of the cubes of the roots.

III. *Solution by* S. FORDE; R. TUCKER, M.A.; *and others.*

Let x, y, z be the roots of the equation

$$t^3+p_1t^2+p_2t+p_3=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13);$$

then, if S_n denote the sum of the nth powers of the roots of (13), we have

$$S_1+p_1=0, \quad S_2+p_1S_1+2p_2=0, \quad S_3+p_1S_2+p_2S_1+3p_3=0;$$

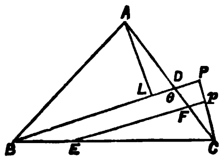
and writing $S_1=a, \; S_2=b^2, \; S_3=c^3$, we find

$$p_1=-a, \quad p_2=\tfrac{1}{2}(a^2-b^2), \quad p_3=\tfrac{1}{6}(-a^3+3ab^2-2c^3).$$

Hence x, y, z can be found by solving the equation (13).

4226. (Proposed by the EDITOR.)—ABC is a triangle, BD a fixed straight line cutting AC at D, and EF a straight line drawn at random parallel to BD. If two points are taken at random within the triangle ABC, prove (1) that the probability of their lying on opposite sides of EF is $\frac{2}{15}\left(2-\frac{UU_1}{\Delta^2}\right)$; where Δ, U, U_1 denote the areas of the triangles ABC, BCD, BDA respectively; (2) that the greatest value of this probability is $\frac{8}{30}$, when BD coincides with AB or BC, and the least value $\frac{7}{30}$, when D is the middle point of AC; and (3) that when the triangle is equilateral, the probability, for a given position of BD, is $\frac{2}{15}+\frac{1}{10}$ cosec^2 BDC, and when the random line EF is drawn across the triangle in an arbitrary manner, the probability is $\frac{2}{15}+\frac{1}{10}\log.3$, or ·243194.

Solution by W. S. B. WOOLHOUSE, F.R.A.S.



1. Draw AL and CP perpendicular to BD; and let BD $=l$, $Cp=x$, CP $=$ X, AL $=X_1$, area CEF $=u$, area ABEF $=u_1$, area BCD $=$ U, area ABD $=U_1$. Then, the ways two points can lie on ABC being Δ^2, the cases in which the points are on opposite sides of EF are $2uu_1$; and for positions of EF between C and BD, observing that u is proportional so x^2, and may be denoted by kx^2, the favorable cases are

$$2\int uu_1\,dx = 2\int u(\Delta-u)\,dx = 2\int kx^2(\Delta-kx^2)\,dx = \tfrac{2}{3}\Delta kx^3-\tfrac{2}{5}k^2x^5$$

$=$ (restoring u for kx^2) $2ux(\frac{1}{3}\Delta-\frac{1}{5}u)$ $=$ (between limits) $2\mathrm{UX}(\frac{1}{3}\Delta-\frac{1}{5}\mathrm{U})$. Substituting $\mathrm{U}+\mathrm{U}_1$ for Δ and $\frac{2\mathrm{U}}{l}$ for X, this is $\frac{4}{15l}(2\mathrm{U}^3+5\mathrm{U}^2\mathrm{U}_1)$; and, by analogy, for positions of EF between A and BD, the cases are $\frac{4}{15l}(2\mathrm{U}_1^3+5\mathrm{U}\mathrm{U}_1^2)$. The total favorable cases are therefore

$$\frac{4}{15l}(2\mathrm{U}^3+5\mathrm{U}^2\mathrm{U}_1+5\mathrm{U}\mathrm{U}_1^2+2\mathrm{U}_1^3)$$

$$=\frac{4}{15l}\left\{2(\mathrm{U}+\mathrm{U}_1)^3-(\mathrm{U}+\mathrm{U}_1)\mathrm{U}\mathrm{U}_1\right\}=\frac{4}{15l}(2\Delta^3-\Delta\mathrm{U}\mathrm{U}_1).$$

The total cases are

$$\int\Delta^2dx=\Delta^2(\mathrm{X}+\mathrm{X}_1)=\Delta^2\quad(\mathrm{U}+\mathrm{U}_1)=\frac{2}{l}\Delta^3;$$

hence, by division, the probability is found to have the value stated in the Question.

2. The maximum value is $\frac{4}{15}$ or $\frac{8}{30}$ when U or $\mathrm{U}_1=0$, that is, when EF is drawn parallel to a side of the triangle. The minimum is $\frac{2}{15}(2-\frac{1}{4})=\frac{7}{30}$ when $\mathrm{U}=\mathrm{U}_1=\frac{1}{2}\Delta$, that is, when BD bisects the side AC.

3. When the triangle is equilateral, if θ denote the angle BDC, then

$$\frac{\mathrm{U}\mathrm{U}_1}{\Delta^2}=\tfrac{1}{4}(1-3\cot^2\theta)\quad\text{and}\quad p=\tfrac{1}{30}(7+3\cot^2\theta)=\tfrac{2}{15}+\tfrac{1}{10}\operatorname{cosec}^2\theta.$$

For every angle θ the number of possible lines is proportional to $\mathrm{X}+\mathrm{X}_1$ or

sin θ; the limits of θ are 60° and 120°; and, the angles of the triangle being alike, it will be sufficient to consider the set of lines BD from the angle B. Therefore, when the random line is drawn across the triangle in an arbitrary manner, the probability is

$$\frac{\int d\theta \sin\theta\,(\frac{2}{15}+\frac{1}{10}\operatorname{cosec}^2\theta)}{\int d\theta\sin\theta}=\frac{2}{15}+\frac{1}{10}\log 3,$$

which is Mr. Watson's result in Question 3693 (*Reprint*, Vol. XVIII., p. 47). In this particular case, from the symmetry of the figure, Mr. Carr's method of solving Question 3693 (*Reprint*, Vol. XX., p. 23), though not strictly accurate, ought obviously to give nearly the same value. Thus, Mr. Watson's true result, as above, is ·243194 ..., that of Mr. Carr being ·243599 ..., which exhibits a very close agreement.

Note on Pythagorean Triangles. *By the Rev.* G. H. Hopkins, M.A.

With reference to the formation of a table of numbers which satisfy the Pythagorean triangle, it is advisable to note that although $p\,(\alpha^2+\beta^2)$, $p\,(\alpha^2-\beta^2)$ and $p\times 2\alpha\beta$ are the general forms, p may itself be a square number, so in reality we arrive at two classes of numbers, namely, $\alpha^2+\beta^2$, $\alpha^2-\beta^2$, $2\alpha\beta$ (where α and β may or may not be primes). For convenience these values may be called the principal values, and the composite numbers may be obtained from these by multiplying each by a common factor which is not a square number. I have drawn attention to this because a table of the principal values can easily be written down according to the following scheme:

	1	4	9	16	25	36	49	64	81	100
1		5	10	17	26	37	50	65	82	101
4			13	20	29	40	53	68	85	104
9				25	34	45	58	73	90	109
16					41	52	65	80	97	116
25						61	74	89	106	125
36							85	100	117	136
49								113	130	149
64									145	164
81										181
100										

The other sides may be obtained from this without difficulty; *e.g.*, $73=8^2+3^2$, $8^2-3^2=55$, the third side may either be found by simple multiplication, $2\times3\times8=48$, or by subtraction, $73-25=48$. It is to be noticed that 25 lies upon the end of the diagonal series of numbers, inclined upwards towards the left hand; as is obvious from the following:

$$2\alpha\beta=\alpha^2+\beta^2-(\alpha-\beta)^2.$$

Of these principal values I have formed tables, similar to the example given, up to 5000, tables also being formed which give the values of the sides as well as the hypotenuse. There are 1904 such principal numbers, of which 1105 is the first that has four principal solutions, viz. 23^2+24^2, 12^2+31^2, 9^2+32^2, 4^2+33^2; the composite solutions being $221\,(1^2+2^2)$, $17\,(1^2+8^2)$, $17\,(4^2+7^2)$, $13\,(2^2+9^2)$, $13\,(6^2+7^2)$, $5\,(10^2+11^2)$, and $5\,(5^2+14^2)$.

NOTE ON PYTHAGOREAN TRIANGLES. *By* H. S. MONCK.

In answering the Question 4102 on the above subject,* I was led to the conclusion that the entire number of possible Pythagorean triangles might be obtained by operating, in the manner therein described, on the elementary triangle whose sides are 3, 5, 4 and its multiples. I communicated this result to the Mathematical Editor of the *Educational Times*, with Question 4177, which was founded upon it. Mr. Glaisher has shown† that my investigation was incorrect, as I had overlooked the possibility of some of the sides becoming negative. I now desire to prove that the entire series of possible Pythagorean triangles can be obtained in the manner specified by introducing negative as well as positive signs, and to point out some other properties of Pythagorean triangles which occurred to me in this investigation.

The method given in Question 4102 for finding Pythagorean triangles, with a given side-difference, is as follows:—Let a and c be the sides, and b the hypotenuse of any one such triangle, another will be given by each of the odd columns in the following series:

$$\begin{array}{llll} a, & a+b, & 2a+2b+c & = \mathrm{A}, \\ b, & a+b+c, & 2a+3b+2c & = \mathrm{B}, \\ c, & b+c, & a+2b+2c & = \mathrm{C}; \end{array}$$

in which the top figure of each column is formed by adding the first and second of the preceding column; the lower figure by adding the second and third, and the intermediate figure is the sum of all three.

If we invert this series, we obtain for the values of a, b, c in terms of A, B, C, the following expressions

$$a = 2\mathrm{A}+\mathrm{C}-2\mathrm{B}, \quad b = 3\mathrm{B}-2\mathrm{A}-2\mathrm{C}, \quad c = 2\mathrm{C}+\mathrm{A}-2\mathrm{B}.$$

It is not difficult to prove that a, b, c are respectively less than A, B, C. For since B is greater than either A or C, 2B is greater than A + C; and on the other hand A + C is greater than B, and, consequently, 2A + 2C is greater than 2B. Hence taking any given Pythagorean triangle, we can obtain another, with the same side-difference, whose sides are algebraically less. I had erroneously assumed that they must also be numerically less; but it is at least evident that they must be so if all six are positive integers.

Carrying on this inverse process then upon any Pythagorean triangle A, B, C and its derivatives, and placing, for the sake of uniformity, the smallest number at the top of the column, we must ultimately arrive at a Pythagorean triangle in which this upper figure shall either

* See p. 20 of this volume. † See p. 54 of this volume.

become 0 or negative. If it becomes 0 it is evident that the column is of the form $\begin{matrix} 0 \\ n, \\ n \end{matrix}$ and this must have been formed from the triangle $\begin{matrix} 3n \\ 5n \\ 4n \end{matrix}$ as is evident by applying the process to it directly.

If the top figure becomes negative, we pass, as Mr. Glaisher remarks, to a new side-difference. I add that we pass to a numerically less one. For if A, B, C are all positive integers, and C be greater than A, a only can be negative. For the value of b is $3B-2A-2C$, and since the maximum value of $A+C$ is $= B\sqrt{2}$, b cannot be less than $(3-2\sqrt{2})B$, which is positive. Again c is positive, for its value is $2C+A-2B$, which will be positive if $\quad 2C+A > 2B$ or $2C+\sqrt{(B^2-C^2)} > 2B$,

i.e., if $\sqrt{(B^2-C^2)} > 2(B-C)$, or $B+C > 4(B-C)$, or $5C > 3B$,

which must be the case if C is the greater side of the triangle. Hence a alone can become negative, and since it is a negative integer and c a positive integer, their arithmetical difference (the new side-difference) is less than their algebraical difference (the old side-difference). The new Pythagorean triangle a, b, c has the same numerical values for its sides as another triangle $-a$, b, c, whose sides are positive integers; and if the latter can be formed from the elementary triangle 3, 5, 4 (introducing negative as well as positive signs), so can the former. Placing then, for the sake of uniformity, whichever c or $-a$ is numerically least at the top of the column, and renewing the inverse process, we arrive at a triangle in which the top figure again becomes either 0 or negative; and again, in the former case we must have passed through some multiple of the triangle 3, 5, 4 at the last preceding stage, while in the latter we pass to a numerically less side-difference. And since we cannot go on lessening (numerically) our side-difference for ever, we must ultimately arrive at a triangle where the upper figure vanishes, and in reaching it we must have passed through the triangle $3n$, $5n$, $4n$.

Let us now see how many really distinct Pythagorean triangles can be formed by introducing negative signs as well as positive, in applying the rule. It is evident that the triangle $-a$, $-b$, $-c$ will give the same derivative triangles as a, b, c, the signs only being changed; and the same thing is evident of $-a$, b, c and a, $-b$, $-c$; of a, $-b$, c and $-a$, b, $-c$; and of a, b, $-c$ and $-a$, $-b$, c. There remain, therefore, four triangles (or rather four methods of formation from the triangle a, b, c) to be considered, viz., a, b, c; a, $-b$, c; $-a$, b, c, and a, b, $-c$. The first gives the series as originally stated in Quest. 4102, and the Pythagorean triangles in this series (the positive signs being retained at every subsequent stage) will have a constant side-difference. Let us next try $-a$, b, c (or a, b, $-c$, since these two will evidently give symmetrical results):

$$\begin{matrix} -a, & b-a, & 2b+c-2a = A', \\ b, & b+c-a, & 3b+2c-2a = B', \\ c, & b+c, & 2b+2c-a = C'. \end{matrix}$$

Here it is easy to verify that $B'^2 = A'^2 + C'^2$, so that a Pythagorean triangle is formed. Also $B'-C' = b-a$, so that in this series (if continued in the same manner) the difference between the hypotenuse and one side will remain constant (that side alternating between the top and the bottom of the column). Lastly, trying a, $-b$, c,

$$\begin{matrix} a, & a-b, & 2a+c-2b = A'', \\ -b, & a+c-b, & 2a+2c-3b = B'', \\ c, & c-b, & 2c+a-2b = C'', \end{matrix}$$

we have evidently the same expressions that are obtained by pursuing the process inversely on a, b, c, except that the sign of B″ is negative; and, in fact, this affords the simplest means of inverting the series. But if we start from the elementary triangle $3n$, $5n$, $4n$, this mode of formation may evidently be neglected, since to introduce it at any stage would be simply to retrace our steps. The other modes of formation are, however, admissible at any stage. Taking any Pythagorean triangle we may have arrived at (no matter by what process), we may take all three figures positively, or treat either the upper only, or the lower only, as negative; and in this way three distinct Pythagorean triangles may be formed from it at the next stage.

Thus, if we follow the rule as laid down in Quest. 4102:—

3	8	20	49	119
5	12	29	70	169
4	9	21	50	120

the first, third, and fifth columns express a Pythagorean triangle. But we obtain two other third columns by taking 3 and 4 respectively negatively, viz.,

3	2	8		3	8	12
5	6	17	and	5	4	13
4	9	15		4	1	5

Taking the three triangles of the third column, viz. 20, 29, 21; 8, 17, 15, and 12, 13, 5, we can again operate on them in each of the three ways, viz.

20	49	119	20	9	39	20	49	77
29	70	169	29	30	89	29	28	85
21	50	120	21	50	80	21	8	36
8	25	65	8	9	33	8	25	35
17	40	97	17	24	65	17	10	37
15	32	72	15	32	56	15	2	12
12	25	55	12	1	7	12	25	45
13	30	73	13	6	25	13	20	53
5	18	48	5	18	24	5	8	28

and, of course, each of the triangles thus arrived at may be operated on similarly.

The three sides of every Pythagorean triangle are always of the form (a and b being positive integers) $2ab$, a^2+b^2, and a^2-b^2 (not implying that the first of these is larger than the third, or *vice versâ*). For it is evident that this is so with the elementary triangle 3, 5, 4, putting $a=2$ and $b=1$; and if it be true of one triangle, it is so of each of the three next. For

$$\begin{array}{lll} 2ab & (a+b)^2 & (2a+b)^2-a^2 = \mathrm{A}, \\ a^2+b^2 & 2a(a+b) & (2a+b)^2+a^2 = \mathrm{B}, \\ a^2-b^2 & 2a^2 & 2a(2a+b) \quad = \mathrm{C}, \end{array}$$

if we take all positively. Taking $2ab$ negatively, we have

$$\begin{array}{lll} 2ab & (a-b)^2 & (2a-b)^2-a^2 = \mathrm{A}', \\ a^2+b^2 & 2a(a-b) & (2a-b)^2+a^2 = \mathrm{B}', \\ a^2-b^2 & 2a^2 & 2a(2a-b) \quad = \mathrm{C}'. \end{array}$$

Taking a^2-b^2 negatively, we have

$$\begin{array}{lll} 2ab & (a+b)^2 & (a+2b)^2-b^2 = \mathrm{A}'', \\ a^2+b^2 & 2b(a+b) & (a+2b)^2+b^2 = \mathrm{B}'', \\ b^2-a^2 & 2b^2 & 2b(a+2b) \quad = \mathrm{C}''; \end{array}$$

which are all of the same form as before. Hence all possible Pythagorean triangles may be found by giving to a and b all possible positive integral values in the expressions $2ab$, a^2+b^2, and a^2-b^2; and, conversely, all such possible values can be obtained from the elementary triangle 3, 5, 4 by the processes already stated. Pythagorean triangles, however, may be formed by any multiples of these figures as well as the figures themselves. The form of the even columns is also worthy of observation. They may always be expressed as $2a^2$, $2ab$, b^2, where a and b are positive integers.

4225. (Proposed by G. M. Minchin, M.A.)—If $q=e^{-\frac{\pi K'}{K}}$, where K is the complete elliptic function of the first kind with modulus k, and K′ the complementary function, show that the limiting value of $\frac{q^n}{k}$ when $k=0$ is zero, unless $n=\frac{1}{2}$, and that the limiting value in this case is $\frac{1}{4}$.

Solution by J. W. L. Glaisher, B.A.

The Question is not quite accurate; the limit is zero if $n>\frac{1}{2}$, $\frac{1}{4}$ if $n=\frac{1}{2}$, and ∞ if $n<\frac{1}{2}$; and the solution is evident at sight from the formula

$$\vartheta = \frac{4\sqrt{q}}{1+q} - \frac{4\sqrt{q^3}}{3(1+q^3)} + \frac{4\sqrt{q^5}}{5(1+q^5)} - \ldots\ldots \text{ (\textit{Fundamenta Nova}, p. 108).}$$

Or otherwise, let p be the limit; then

$$\begin{aligned}\log p &= n\log q - \log k \\ &= (n-\tfrac{1}{2})\log q - \log 4 + 4q - 6q^2 - \ldots\ldots \text{ (\textit{Fund. Nov.}, p. 104),}\end{aligned}$$

and the same conclusions follow.

4121. (Proposed by Professor Townsend, M.A., F.R.S.)—The curve of free equilibrium of a uniform flexible cord, under the action of a central repulsive force varying as any function of the distance, being supposed confined to the narrow interval between two circles of nearly equal radii concentric with each other and with the centre of force; determine approximately the angle, real or imaginary, between its apsides.

Solution by the Proposer.

Denoting, as in the corresponding problem for the free motion of a material particle under the action of a central force, by u and θ the reciprocal of the radius vector and the polar angle respectively, by μ the absolute force, and by $u^2.\phi(u)$ the law of force; the differential equation of the curve of equilibrium of the cord, supposed entirely unrestricted, is easily seen to be

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu^2\epsilon^2}{H^2}\cdot\phi'(u)\cdot\phi(u) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where ϵ = the constant mass per unit of length of the cord, H the constant product of the tension T into the perpendicular p on the tangent from the centre of force throughout its entire length, and $\phi'(u) = \int \phi(u)\,du$, an arbitrary constant being of course involved in the last.

Putting now, in this exact equation, $u = c + z$, where c = the mean value of u, that viz. for which $c - \frac{\mu^2 \epsilon^2}{H^2} \cdot \phi'(c) \cdot \phi(c) = 0$; substituting for $\phi(u)$ and $\phi'(u)$ their approximate values to the first power of the small fluctuating quantity z, viz., $\phi(c) + \phi'(c) \cdot z$ and $\phi'(c) + \phi(c) \cdot z$; and rejecting in the result all powers of z higher than the first; we obtain for z the approximate equation

$$\frac{d^2 z}{dt^2} + \left[1 - c\left(\frac{\phi(c)}{\phi_1(c)} + \frac{\phi'(c)}{\phi(c)}\right)\right] \cdot z = 0 \quad \text{.................} (2),$$

or [remembering that, while the mean value of T in the actual cord $= \mu\epsilon \cdot \phi'(c)$, its exact value, were c absolutely invariable, would be $\mu\epsilon \cdot c \cdot \phi(c)$, so that approximately $\phi'(c) = c \cdot \phi(c)$] the simplified equation

$$\frac{d^2 z}{dt^2} - c \cdot \frac{\phi'(c)}{\phi(c)} \cdot z = 0 \quad \text{..........................} (3),$$

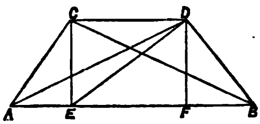
from which it appears at once that the required angle, real or imaginary, between the apsides

$$= \frac{\pi}{\left[-c \cdot \frac{\phi'(c)}{\phi(c)}\right]^{\frac{1}{2}}} \quad \text{..............................} (4).$$

Cor. If $F = \mu \cdot u^n = \mu \cdot u^2 \cdot u^{n-2}$; since then $c \cdot \frac{\phi'(c)}{\phi(c)} = n - 2$, therefore for a real apsidal angle $n-2$ must not exceed 0, or n must not exceed 2. Hence, for a force varying inversely as any simple power (n) of the distance, the apsidal angle is infinite for $n = 2$, and imaginary for $n > 2$.

3665. (Proposed by Dr. Hart.)—Find a trapezoid whose sides, diagonals, perpendicular between the parallels, and area, shall be integral numbers.

Solution by the Proposer.



Let $AE = m^2 - n^2$, $CE = DF = 2mn$
$AF = n(m^2 - 1)$, $BF = m(n^2 - 1)$; then
$AB = n(m^2-1) + m(n^2-1) = (mn-1)(m+n)$,
$EF = CD = (m^2+n)(n-1)$, $AC = m^2 + n^2$,
$BD = m(n^2+1)$, $AD = n(m^2+1)$,
$BE = (m+1)(m+n)(n-1)$, therefore

$$BC^2 = (m+1)^2(m+n)^2(n-1)^2 + 4m^2n^2 = (m+1)^2 n^4 + 2(m+1)^2(m-1)n^3$$
$$+ (m^4 - 2m^3 - 2m^2 - 2m + 1)n^2 - 2m(m+1)^2(m-1)n + m^2(m+1)^2 = \square$$
$$= (m+1)^2\left\{n^2 - (m-1)n + m\right\}^2, \text{ suppose, then } n = \frac{m(m^2+m+1)}{(m-1)(m+1)^2},$$

where m may be any number > 1. Let $m = 2$, then we have

$$n = \tfrac{14}{9},\ AB = \tfrac{608}{81},\ AC = \tfrac{520}{81},\ CD = \tfrac{250}{81},\ BD = \tfrac{554}{81},\ AD = \tfrac{70}{9},$$
$$BC = (m+1)\left\{n^2-(m-1)\,n+m\right\} = \tfrac{232}{27},\quad CE = \tfrac{56}{9};$$

or, multiplying all these quantities by 81, we have $AB = 608$, $AC = 520$, $CD = 250$, $BD = 554$, $AD = 630$, $BC = 696$, $CE = 504$, Area $= 216216$.

When the trapezoid has one of its sides, as CE, perpendicular to the parallels, we may proceed in a similar manner.

When $AE = BF$, then $AF = BE$, $AC = BD$, and $AD = BC$; let $AE = BF = n(m^2-1)$, $AF = BE = m(n^2-1)$, and $CE = DF = 2mn$, then $AC = BD = n(m^2+1)$, $CD = EF = m(n^2-1) - n(m^2-1) = (mn-1)(m-n)$, $AD = BC = m(n^2+1)$, and $AB = (mn-1)(m+n)$.

In a similar manner we may find the sides, diagonals, perpendicular, and area of any parallelogram except a square.

3641. (Proposed by Dr. HART.)—Find three right-angled triangles whose perimeters shall be equal, and whose areas shall be in arithmetical progression.

I. *Solution by the* Rev. U. JESSE KNISELY.

Represent the sides as follows:

1.	$(p^2+2pq)\,a$,	$(2pq+2q^2)\,a$,	$(p^2+2pq+2q^2)\,a$.
2.	$(p^2-q^2)\,b$,	$2pqb$,	$(p^2+q^2)\,b$.
3.	$(p^2-4q^2)\,c$,	$4pqc$,	$(p^2+4q^2)\,c$.

	Areas.	Perimeters.	Radii of inscribed circles.
1.	$(p^2+2pq)(pq+q^2)\,a^2$,	$2\,(p^2+3pq+2q^2)\,a$,	pqa.
2.	$pq\,(p^2-q^2)\,b^2$,	$(2p^2+2pq)\,b$,	$q\,(p-q)\,b$.
3.	$2pq\,(p^2-4q^2)\,c^2$,	$(2p^2+2pq)\,c$,	$2q\,(p-2q)\,c$.

By the first condition we have

$$\left.\begin{array}{l} \text{from 1 and 2, } (p^2+3pq+2q^2)a = (p^2+pq)\,b, \text{ or } (p+2q)a = pb \ldots (1);\\ \text{from 1 and 3, } (p^2+3pq+2q^2)\,a = (p^2+2pq)\,c, \text{ or } (p+q)\,a = pc \ldots (2).\\ \text{By the second condition, } 2\,(p-q)\,b = pa+2pc-4qc \ldots\ldots\ldots\ldots\ldots (3). \end{array}\right\}\ (A).$$

From (1), $b = \dfrac{(p+2q)\,a}{p}$, and from (2), $c = \dfrac{(p+q)\,a}{p}$.

Substituting in (3) and dividing by a, we have

$$\frac{2\,(p-q)\,(p+2q)}{p} = p + 2\,(p+q) - \frac{4pq+4q^2}{p};\ \text{whence } p = 4q.$$

Hence $b = \frac{6}{4}a$, $c = \frac{5}{4}a$, and q and a may be assumed at pleasure. If q be taken $= 1$, $p = 4$; if a be taken $= 4$, to avoid fractions, $c = 5$, $b = 6$.

Then substituting in the original assumptions, we have the triangles,

$$\left.\begin{array}{l} 96,\ 40,\ 104\\ 90,\ 48,\ 102\\ 80,\ 60,\ 100 \end{array}\right\};\ \text{or, dividing by 2,}\quad \left.\begin{array}{l} 20,\ 48,\ 52\\ 24,\ 45,\ 51\\ 30,\ 40,\ 50 \end{array}\right\}.$$

II. *Solution by* ASHER B. EVANS, M.A.

Let $2amn$, $a(m^2-n^2)$, $a(m^2+n^2)$; $2bpq$, $b(p^2-q^2)$, $b(p^2+q^2)$; $2crs$, $c(r^2-s^2)$, $c(r^2+s^2)$, represent the sides of the three required triangles; then will $2a(mn+m^2)$, $2b(pq+p^2)$, $2c(rs+r^2)$ represent their perimeters, and $a(mn-n^2)$, $b(pq-q^2)$, $c(rs-s^2)$ the radii of their inscribed circles. The perimeters of three triangles being equal, their areas will be in arithmetical progression if the radii of their inscribed circles are in arithmetical progression; the conditions of the question will therefore be satisfied when

$$a(mn+m^2) = b(pq+p^2) = c(rs+r^2) \quad \text{.................... (1)},$$
$$a(mn-n^2) + c(rs-s^2) = 2b(pq-q^2) \quad \text{.................. (2)}.$$

By eliminating a, b, c, between (1) and (2), we have

$$\frac{mn-n^2}{mn+m^2} + \frac{rs-s^2}{rs+r^2} = 2\left(\frac{pq-q^2}{pq+p^2}\right) \quad \text{.................... (3)}.$$

Let
$$\theta = \frac{mn-n^2}{mn+m^2} \quad \text{.............................. (4)};$$

then, from (3), we obtain

$$p^2\left\{r^2\theta+rs(\theta+1)-s^2\right\}+pq\left\{r^2(\theta-2)+rs(\theta-1)-s^2\right\} = 2q^2(rs+r^2);$$

which, the square being completed and the root extracted, becomes

$$2p\left\{r^2\theta+rs(\theta+1)-s^2\right\}+q\left\{r^2(\theta-2)+rs(\theta-1)-s^2\right\} = \mathrm{A}q \quad \text{... (5)};$$

where

$$\mathrm{A}^2 = r^4(\theta^2-12\theta+4)+2r^3s(\theta^2-11\theta-2)+r^2s^2(\theta^2-12\theta+5)-2rs^3(\theta-5)+s^4 \quad \text{...... (6)}.$$

Assume
$$\mathrm{A} = s^2-rs(\theta-5)-r^2(\theta+10);$$

then from (6), we have
$$\frac{s}{r} = \frac{3+\theta}{3-\theta}.$$

In order to make $r > s$, while θ is positive, let us take

$$r=\beta(3-\theta) \quad\text{and}\quad s=\beta(3+\theta-x) \quad \text{.................. (7)};$$

then from (6) we find

$$\mathrm{A} = \beta^2\left\{x^4-2(21-6\theta+\theta^2)x^3+(369-156\theta+62\theta^2-12\theta^3+\theta^4)x^2 - 8(135-126\theta+93\theta^2-24\theta^3+2\theta^4)x+16(9-12\theta+2\theta^2)^2\right\}^{\frac{1}{2}} \quad \text{...... (8)}.$$

Assume A in (8) equal to

$$\beta^2\left\{-x^2+(21-6\theta+\theta^2)x-4(9-12\theta+2\theta^2)\right\} \quad \text{............ (9)};$$

then from (8) and (9), we have

$$(9-12\theta+2\theta^2)x = 27-90\theta+15\theta^2 \quad \text{.................. (10)}.$$

From (5), by the aid of (9), we find

$$2p(12\theta-8\theta^2+3x+\theta^2x-x^2) = 2q(12\theta-2\theta x+6x);$$

which, by the aid of (10), reduces to

$$p(27-81\theta+21\theta^2-\theta^3)(45\theta-15\theta^2+\theta^3)$$
$$= 3q(27-81\theta+21\theta^2-\theta^3)(9-12\theta+2\theta^2),$$

or
$$p(45\theta-15\theta^2+\theta^3) = 3q(9-12\theta+2\theta^2).$$

Let $p = 3\gamma(9-12\theta+2\theta^2)$, then $q = \gamma(45\theta-15\theta^2+\theta^3)$ .... (11).

From (7) and (10), by putting $\beta=\delta(9-12\theta+2\theta^2)$, we find

$$\left.\begin{aligned} r &= \delta(27-45\theta+18\theta^2-2\theta^3) \\ s &= \delta(63\theta-21\theta^2+2\theta^3) \end{aligned}\right\} \quad \text{..................... (12)}.$$

Equations (1), (4), (11) and (12) furnish a complete general solution to the question, the quantities m, n, γ, δ being assumed at pleasure, provided that m is greater than n.

For a numerical example, let $m=2$, $n=1$, $\gamma=216$, $\delta=108$, then we have $\theta=\frac{1}{8}$, $p=4572$, $q=1531$, $r=2159$, $s=1072$, and from (1) we obtain

$$6a = 27202916b = 6975729c;\text{ whence } a = 4650486b, \text{ and } c = 4b.$$

The sides of the three triangles are thus found to be

$$\begin{aligned} 2amn &= 18601944b, & a(m^2-n^2) &= 13951458b, & a(m^2+n^2) &= 23252430b,\\ 2bpq &= 13999464b, & b(p^2-q^2) &= 18559223b, & b(p^2+q^2) &= 23247145b,\\ 2crs &= 18515584b, & c(r^2-s^2) &= 14048388b, & c(r^2+s^2) &= 23241860b; \end{aligned}$$

where b may take any positive integral value.

4020. (Proposed by Dr. HART.)—Find any number of square numbers whose sum shall be a square.

I. *Solution by* ARTEMAS MARTIN.

Let $x_1^2, x_2^2, x_3^2, \dots x_{n-1}^2, x_n^2$ be n square numbers.

Then must $x_1^2+x_2^2+x_3^2+\dots+x_{n-1}^2+x_n^2 = \square = (x_n+a)^2$, suppose;

then we have
$$x_n = \frac{x_1^2+x_2^2+x_3^2+\dots+x_{n-1}^2-a^2}{2a}.$$

Reducing to a common denominator, and then discarding it, the roots of the n squares are $2ax_1$, $2ax_2$, $2ax_3$, ... $2ax_{n-1}$ and $x_1^2+x_2^2+\dots+x_{n-1}^2-a^2$, where $x_1, x_2, x_3, \dots x_{n-1}$, and a may have any values chosen at pleasure.

Let $n=2$, then the roots of two squares whose sum is a square, are $2ax_1$ and x^2-a^2, which are well known expressions.

Let $n=3$, then the roots of three squares whose sum is a square, are $2ax_1$, $2ax_2$ and $x_1^2+x_2^2-a^2$.

Take $a=1$, $x_1=2$, $x_2=3$; then we have $2^2+3^2+6^2=7^2$.

II. *Solution by the* PROPOSER.

Let $2a\text{N}$, $2b\text{N}$, $2c\text{N}$, $2d\text{N}$, &c. to $(n-1)$ numbers be the roots of $n-1$ squares, and let $\{a^2+b^2+c^2+d^2+\text{\&c. to } (n-1) \text{ numbers}-\text{N}^2\}$ be the root of the remaining square. Then

$$4(a^2+b^2+c^2+d^2+\dots\text{to } n-1 \text{ nos.})\,\text{N}^2+(a^2+b^2+c^2+d^2+\dots\text{to } n-1 \text{ nos.}-\text{N}^2)$$
$$= (a^2+b^2+c^2+d^2\dots\text{to } n-1 \text{ nos.}+\text{N}^2)^2 = \square.$$

There are many other ways which may be employed in solving this problem, but the above is, perhaps, the most simple.

III. *Solution by* H. S. MONCK.

Let a^2 be equal to n square numbers, and let b^2 be a square number, then $(a^2+b^2)^2$ is a square number made up of $n+1$ square numbers.

For let $a^2 = (b+c)^2$; then $(a^2+b^2)^2 = \left\{b^2+(b+c)^2\right\}^2$

$$= 4b^4+8b^3c+8b^2c^2+4bc^3+c^4 = 4b^2(b+c)^2+(2bc+c^2)^2.$$

Now if $$(b+c)^2 = \alpha^2+\beta^2+\gamma^2+\delta^2+\&c.,$$
we obviously have

$$(a^2+b^2)^2 = 4b^2\alpha^2+4b^2\beta^2+4b^2\gamma^2+4b^2\delta^2+\ldots+(2bc+c^2)^2.$$

Each of these is a perfect square, and their number is $n+1$, since $(2bc+c^2)^2$ does not occur in the previous expression. Its value is $(a^2-b^2)^2$.

Hence it is plain that, starting from any hypotenuse of a Pythagorean triangle* [for the hypotenuse, as well as its square, is always equal to the sum of two squares, as appears from the expression given for it on pp. 20, 21 of this volume, viz., $2a^2+2ab+b^2 = a^2+(a+b)^2$] we can go on to form a square number which is equal to the sum of any number of squares.

If $n-1$ (n being the required number of squares) be a composite number, a shorter method may be taken. Let $n-1 = lm$. Then, if a^2 be a square number whose sum is $=l$ square numbers, and b^2 be another whose sum is equal to m squares, $(a^2+b^2)^2$ will be resolvable into n square numbers.

A continuation of this series may be obtained by the method given in Question 4102.† Thus, if $(b+c)^2$ be equal to the sum of n squares, and b^2 be a square number, we can form the Pythagorean triangle and next terms as follows :—

$$2b(b+c),\quad (2b+c)^2,\quad (4b+3c)(2b+c),\quad (b+c)^2+b^2,\quad 2(b+c)(2b+c),$$
$$(3b+2c)^2+(b+c)^2,\quad 2bc+c^2,\quad 2(b+c)^2,\quad 2(3b+2c)(b+c).$$

Now, since $(b+c)^2$ = the sum of n squares, $\left\{(3b+2c)^2+(b+c)^2\right\}^2$ is the sum of $n+1$ squares. However, beyond proving a curious property of these triangles, the result is of no value, as the requisite numbers are more easily obtained otherwise.

3976. (Proposed by Dr. HART.)—Find general values of x, y, z that will make $x^2+y^2-z^2 = \square$, $x^2+z^2-y^2 = \square$, $y^2+z^2-x^2 = \square$.

I. *Solution by* SAMUEL BILLS.

Put $y^2+z^2-x^2 = h^2,\quad x^2+z^2-y^2 = k^2,\quad x^2+y^2-z^2 = l^2 \;\ldots\ldots\; (1, 2, 3).$

Adding, we have $h^2+k^2 = 2z^2,\quad h^2+l^2 = 2y^2,\quad k^2+l^2 = 2x^2 \;\ldots\ldots\; (4, 5, 6).$

Assume $$\pm k = \frac{p^2-q^2+2pq}{p^2-q^2-2pq}\,h,\qquad \pm l = \frac{r^2-s^2+2rs}{r^2-s^2-2rs}\,h;$$

then (4) and (5) will be satisfied, and we shall find

$$z = \frac{p^2+q^2}{p^2-q^2-2pq}\,h, \text{ and } y = \frac{r^2+s^2}{r^2-s^2-2rs}\,h.$$

It remains to find $$\left(\frac{p^2-q^2+2pq}{p^2-q^2-2pq}\right)^2+\left(\frac{r^2-s^2+2rs}{r^2-s^2-2rs}\right)^2 = 2\,\frac{x^2}{h^2},$$

* See p. 54 of this volume. † See p. 20 of this volume.

or, by developing the above, and dividing by 2, we shall have to find

$$p^4 - 16\,\frac{rs\,(r^5 - s^2)}{(r^2+s^2)^2}\,p^3q + 2p^2q^2 + 16\,\frac{rs\,(r^2-s^2)}{(r^2+s^2)^2} + q^4 = \square$$

$$= \text{(suppose)}\left(p^2 - 8\,\frac{rs\,(r^2-s^2)}{(r^2+s^2)^2}\,pq + q^2\right)^2; \quad \text{whence} \quad p = \frac{(r^2+s^2)^2}{2rs\,(r^2-s^2)}\,q,$$

which gives a very general solution to the question.

Taking $r=2$ and $s=1$, we find $p = \dfrac{25}{12}q$; we may therefore take $p=25$ and $q=12$; which gives $k = \dfrac{1081}{119}h$, $l=7h$; take $h=119$, then $k=1081$ and $l=833$. We find $z=769$, $y=595$, $x=965$.

II. *Solution by the* Proposer.

Let $x^2+y^2-z^2 = m^2$, $x^2+z^2-y^2 = n^2$, $y^2+z^2-x^2 = p^2$; then adding the first and second, first and third, and second and third, and multiplying the respective sums by 2, we have

$$2m^2+2n^2 = 4x^2 = \square, \quad 2m^2+2p^2 = 4y^2 = \square, \quad 2n^2+2p^2 = 4z^2 = \square.$$

Now $2m^2+2n^2 = \square$, when $m = r^2-2s^2$, and $n = r^2+4rs+2s^2$; hence

$$2m^2+2p^2 = 2\,(r^2-2s^2)^2 + 2p^2 = \square \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$2n^2+2p^2 = 2\,(r^2+4rs+2s^2)^2 + 2p^2 = \square \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

(1) is a square, if $p = r^2-2s^2$; let $p = q+r^2-2s^2$, then we have

$$(1) = 2q^2 + 4\,(r^2-2s^2)\,q + 4\,(r^2-2s^2)^2 = \square,$$

$$(2) = 2q^2 + 4\,(r^2-2s^2)\,q + 4\,(r^2+2rs+2s^2)^2 = \square;$$

or putting $a = r^2+2rs+2s^2$, $m = r^2-2s^2$, we have

$$2q^2+4mq+4m^2 = \square \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$2q^2+4mq+4a^2 = \square \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

Let $(4) = (2a-qt)^2$, then we have $q = \dfrac{4\,(at+m)}{t^2-2}$; substituting this value of q in (3), multiplying by $\left(\dfrac{t^2-2}{2}\right)^2$, and arranging the terms, we have

$$m^2t^4 + 4amt^3 + 8a^2t^2 + 8amt + 4m^2 = \square = (mt^2-2at-2m)^2,$$

whence we find $\quad t = -\dfrac{a^2+m^2}{2am}, \quad q = -\dfrac{8a^2m\,(a^2-m^2)}{a^2-6a^2m^2+m^2};$

therefore $\quad p = \dfrac{m\,(m^4+2a^2m^2-7a^4)}{a^4-6a^2m^2+m^4}.$

In the values of m, n, a above, r, s may be any numbers whatever. Let $r=1$, $s=1$; then $m=-1$; $n=7$, $a=5$, whence $p = -\frac{1081}{119}$; then we have

$$4x=100, \quad x^2=25, \quad 4y^2=\left(\tfrac{1538}{119}\right)^2, \quad y^2=\left(\tfrac{769}{119}\right)^2, \quad 4z^2=\left(\tfrac{1930}{119}\right)^2, \quad z^2=\left(\tfrac{965}{119}\right)^2;$$

or, multiplying by $(119)^2$, for the required squares are found to be

$$(595)^2,\ (769)^2,\ (965)^2.$$

By a method resembling the above, I get, by a particular solution, $x^2=(149)^2$, $y^2=(241)^2$, $z^2=(269)^2$, which have long been known as the least numbers that will satisfy the conditions of the Question.

The general values of x, y, z will be

$$x=\tfrac{1}{2}(2m^2+2n^2)^{\frac{1}{2}},\quad y=\tfrac{1}{2}(2m^2+2p^2)^{\frac{1}{2}},\quad z=\tfrac{1}{2}(2n^2+2p^2)^{\frac{1}{2}}.$$

If the general values of m, n, p be substituted in the above values of x, y, z, we shall have rational expressions for these quantities, but they will be rather complicated.

3726. (Proposed by Dr. HART.)—Find n numbers such that if the square of the first be added to the second, the square of the second to the third, the square of the third to the fourth, ... and the square of the nth to the first, the respective sums shall all be squares.

I. *Solution by* ASHER B. EVANS, M.A.

Let $a_1, a_2, a_3, \ldots a_n$ represent the n numbers. The first $n-1$ of the required n conditions will be satisfied if we make

$$a_r = m^2+2ma_{r-1} \quad\text{.............................}\quad (1);$$

in which r takes in succession all integral values from 2 to n. Since by (1)

$$a_2 = m^2+2ma_1,\quad a_3 = m^2+2ma_2,\ \ldots\ldots\ a_n = m^2+2ma_{n-1},$$

we have $\quad a_n = m^2+2m^3+2^2m^4+\ \ldots\ldots\ +2^{n-2}m^n+2^{n-1}m^{n-1}a_1$ .... (2).

By putting $\quad m^2+2m^3+2m^4+\ \ldots\ +2^{n-2}m^n = A$ .... (3),

we may write (2) $\quad a_n = A+2^{n-1}m^{n-1}a_1$ .... (4).

But $\quad a_n^2+a_1 = A^2+2^nm^{n-1}a_1A+a_1+2^{2n-2}m^{2n-2}a_1^2$

must be a square; assume $(2^{n-1}m^{n-1}a_1+p)$ for the root of this square; then will $2^{2n-2}m^{2n-2}a_1^2+(2^nm^{n-1}A+1)a_1+A^2 = (2^{n-1}m^{n-1}a_1+p)^2$;

and therefore $\quad a_1 = \dfrac{p^2-A^2}{2^nm^{n-1}(A-p)+1}$ .... (5).

By taking p and m any positive numbers at pleasure, provided $p>A$ and $<A+\dfrac{1}{2^nm^{n-1}}$, equations (1), (3) and (5) will furnish a complete solution to the question.

II. *Solution by the* PROPOSER.

(1.) For two numbers, x, y, then $x^2+y=\square$, and $y^2+x=\square$. The first expression is a square when $y=2x+1$. (2.) For three numbers, x, y, z, then $x^2+y=\square$, $y^2+z=\square$, and $z^2+x=\square$. The first expression is a square when $y=2x+1$, and the second when $z=4x+3$. (3.) For four numbers, x, y, z, w, then $x^2+y=\square$, $y^2+z=\square$, $z^2+w=\square$, and $w^2+x=\square$. The first expression is a square when $y=2x+1$, the second when $z=4x+3$, and the third when $w=8x+7$. In the same way, for five numbers, we shall find the fourth expression a square when the fifth number is $16x+15$,

&c. We see now that the coefficient of the first term in the respective assumptions for y, z, w, &c., is a power of 2, and that the second term is the same power of 2 less 1; also, that the exponent of 2 is one less than the number of quantities required; therefore the assumption for the nth number is $2^{n-1}x+2^{n-1}-1$, the square of which $+x$ is to be a square,

or $$2^{2(n-1)}x^2+(2^{2n-1}-2^n+1)\,x+(2^{n-1}-1)^2=\square.$$

Suppose this to be equal to $\left(\frac{p}{q}x-(2^{n-1}-1)\right)^2$.

Reducing, we find $$x=\frac{(2^n-2)\,pq+(2^{2n-1}-2^n+1)\,q^2}{p^2-2^{2(n-1)}q^2},$$

whence the remaining quantities y, z, w, &c. are known, n being any number >1, and p, q any numbers that will make $p^2>2^{2(n-1)}q^2$.

A PYTHAGOREAN SIEVE. *By* J. W. L. GLAISHER, B.A.

In connexion with the subject of Pythagorean triangles I may notice the following scheme, the mode of formation of which is evident, and which may be called a Pythagorean sieve:—

1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	...
−3,	−6,	−9,	−12,	−15	−18,	−21,	−24,	−27,	−30,	−33,	−36,	−39,	−42,	...
5,	10,	15,	20,	25,	30.	35,	40,	45,	50,	55,	60,	65,	70,	...
−7,	−14,	−21.	−28,	−35,	−42,	−49,	−56,	−63,	−70,	−77,	−84,	−91,	−98,	...
9,	18,	27,	36,	45,	54,	63,	72,	81,	90,	99,	108,	117,	126,	...
−11,	−22,	−33,	−44,	−55,	−66,	−77,	−88,	−99,	−110,	−121,	−132,	−143,	−154,	...
13,	26,	39,	52,	65,	78,	91,	104,	117,	130,	143,	156,	169,	182,	...
−15,	−30,	−45,	−60,	−75,	−90,	−105,	−120,	−135,	−150,	−165,	−180,	−195,	−210,	...
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

Strike out all the numbers that cancel one another, and every number that remains is expressible as the sum of two squares (0 being regarded as a square). The converse proposition is also true; viz., that every number that can be decomposed into the sum of two squares will remain. Thus $1=1^2+0^2$; $2=1^2+1^2$; 3 is cancelled; $4=2^2+0^2$; $5=2^2+1^2$; 6 and 7 are cancelled; $8=2^2+2^2$; 9 is cancelled in the 3-line, but reappears in the 9-line, so that it remains, as it ought to do, since $9=3^2+0^2$; $10=3^2+1^2$, 11 and 12 are cancelled; $13=3^2+2^2$, &c. In general if a number occurs in the sieve r times with a positive sign, and s times with a negative sign, then it is retained unless $r=s$; s can never be greater than r. Of course, if 0 be not regarded as a number, the theorem runs, "is either a square, or expressible as the sum of two squares."

The sieve for numbers decomposable into the sum of two *odd* squares is as follows:—

2,	6,	10,	14,	18,	22,	26,	30,	34,	38,	42,	...
−6,	−18,	−30,	−42,	−54,	−66,	−78,	−90,	−102,	−114,	−126,	...
10,	30,	50,	70,	90,	110,	130,	150,	170,	190,	210,	...
−14,	−42,	−70,	−98,	−126,	−154,	−182,	−210,	−238,	−266,	−294,	...
...	...	...	...	...	...	...	...	...	...	...	...

in which every number that remains after the cancelling is the sum of two odd squares; and every such number remains.

The first sieve follows intuitively from the formulæ

$$\left(\frac{2K}{\pi}\right)^{\frac{1}{2}} = 1 + 2\left\{q + q^4 + q^9 + q^{16} + \dots\right\},$$

$$\frac{2K}{\pi} = 1 + 4\left\{\frac{q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \dots\right\},$$

by equating the square of the former to the latter. The second is deduced in the same way from the corresponding series for $\left(\frac{2kK}{\pi}\right)^{\frac{1}{2}}$ and $\frac{2kK}{\pi}$.

I have not attempted to prove them in a purely arithmetical manner, and I imagine it would not be easy to do so. A great number of other theorems relating to decomposition into sums of squares can be deduced from the formulæ in the *Fundamenta Nova*, but most of these have been already published by Jacobi, Eisenstein, and Professor H. J. S. Smith. The two sieves given above are taken from a paper of mine, "On Certain Propositions in the Theory of Numbers deduced from Elliptic-Transcendent Identities," read before the British Association at Bradford.

The sieves of course are only of theoretical interest, and afford a very bad method indeed of finding practically the numbers that are the sums of two (or two odd) squares. To construct such a table, the easiest method would be to take a table of squares and add the lowest square to all above it; then the next square to all above it, and so on to the limit of the table, and finally to arrange the numbers with their decompositions in order. This was, I believe, the method actually employed by Mr. S. M. Drach in the formation of two very large MS. tables showing the decompositions into sums of two or three squares of every number so decomposable, up to a very high number, which were presented by him to the Royal Society, and are now in their library; but I write from memory after a hurried examination of the tables more than a year ago, and I cannot at this moment conveniently verify my statement.

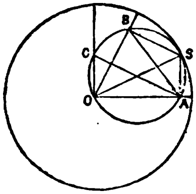
4236. (Proposed by Professor CLIFFORD.)—C is the double point of a circular cubic, and a straight line cuts the curve in D, E, F; join CD, CE, CF, and on the two latter lines take A, B, so that CA . CE = CB . CF; then prove that AB and CD are equally inclined to the tangents at C.

Solution by J. J. WALKER, M.A.

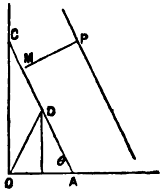
Let D′, E′, F′ be the inverse points of D, E, F respectively, with respect to C as pole; then C, D′, E′, F′ are four points in which a circle (the inverse of the straight line DEF) meets a conic (the inverse of the given cubic) whose asymptotes are parallel to the tangents of the cubic at C. The lines CD, E′F′ are therefore equally inclined to those tangents; but AB is plainly parallel to E′F′.

4068. (Proposed by Professor Wolstenholme, M.A.)—**A lamina moves in its own plane in such a way that two points fixed in the lamina describe two straight lines fixed in space; prove (1) that the motion may be completely represented by supposing a circle fixed in the lamina to roll within a circle of twice the radius fixed in space; (2) that each point of the lamina describes an ellipse with its centre at the centre of the fixed circle; (3) that the foci of these ellipses for a system of points lying in a straight line lie on a rectangular hyperbola; and (4) that every straight line fixed in the lamina envelopes an involute of a four-cusped hypocycloid.**

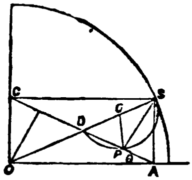
I. *Solution by the* Proposer.



(Fig. 1.)



(Fig. 2.)



(Fig. 3.)

Let OA, OB (Fig. 1) be the two fixed straight lines, A, B the two points of the lamina; then AB is a constant length. Draw AS, BS at right angles to OA, OB; then, since AB and the angle BOA are fixed, the circle OASB is of constant magnitude and will be a fixed circle in the lamina and may be supposed rigidly connected with it; hence also OS, the diameter of this circle, is of constant length, and S will trace out a circle fixed in space of twice the radius of the former circle. But S is the centre of instantaneous rotation of the lamina, hence the smaller circle may be imagined to roll on the larger. Draw OC at right angles to AO meeting the moving circle in C; then CA is of constant length, and the angle BAC = BOC is fixed; whence C is a fixed point in the lamina, and we may replace A, B by A, C, or suppose the two fixed straight lines to be at right angles to each other, without loss of generality. Also any straight line may be supposed to be parallel to AC, for it will be parallel to some diameter of the moving circle, and the ends of this diameter will move on two fixed straight lines at right angles to each other.

Let $\angle$ OAC = θ (Fig. 2), D the centre of AC and of the moving circle, AC = $2a$, DM = x, MP = y, coordinates of any point P of the lamina referred to axes through D along and perpendicular to AC. Then if (X, Y) be the coordinates of P referred to OA, OC as axes,

$$X = a\cos\theta - x\cos\theta + y\sin\theta = (a-x)\cos\theta + y\sin\theta,$$
$$Y = a\sin\theta + x\sin\theta + y\cos\theta = y\cos\theta + (a+x)\sin\theta;$$

therefore $$X(a+x) - Yy = (a^2 - x^2 - y^2)\cos\theta,$$

and $$Xy - Y(a-x) = (y^2 + x^2 - a^2)\sin\theta;$$

hence the locus of P in space is

$$\left\{X(a+x) - Yy\right\}^2 + \left\{Xy - Y(a-x)\right\}^2 = (x^2+y^2-a^2)^2,$$

or $$X^2\left\{(a+x)^2 + y^2\right\} - 4XYay + Y^2\left\{y^2 + (a-x)^2\right\} = (x^2+y^2-a^2)^2,$$

an ellipse whose centre is O, which reduces to a circle only for the point

D ($x=0$, $y=0$), and is a straight line if $x^2+y^2=a^2$, that is, for any point on the moving circle. The foci of this ellipse are given by

$$\frac{X^2-Y^2}{4ax}=\frac{XY}{-2ay}=\frac{(x^2+y^2-a^2)^2}{4a^2y^2-(x^2+y^2+a^2)^2+4a^2x^2}=-1;$$

hence, if either x or y be given, the foci lie on a fixed rectangular hyperbola. If $x^2+y^2=c^2$, $(X^2-Y^2)^2+4X^2Y^2=16a^2c^2$, or $X^2+Y^2=4ac$, that is, if a series of points lie on a circle in the lamina concentric with the moving circle, the foci of the paths also lie on a circle.

Next, to find the envelope of any straight line fixed in the lamina, take a straight line parallel to AC, and if we find the evolute of the envelope of AC, the straight line will envelope a parallel to the curve enveloped by AC, and therefore a curve having the same evolute. Draw (Fig. 3) SP perpendicular to AC; then, since S is the centre of instantaneous rotation, P is the point of contact of AC with its envelope; on SD describe a circle, centre O, this will pass through P, and

$$\text{the angle DOP} = \text{twice} \angle \text{DSP} = 2\text{DSA}-2\text{PSA}$$

$$=\pi-2\theta-2\theta=\pi-4\theta, \text{ therefore } \angle\text{POS}=4\theta=4\angle\text{SOA};$$

hence the arc SP = arc of the fixed circle measured from OA, or P traces a four-cusped hypocycloid touching AC at P. The evolute of this is a four-cusped hypocycloid of twice the linear dimensions, and therefore generated by a circle of radius a rolling within one of radius $4a$, and any straight line fixed in the lamina will touch some involute of such an hypocycloid. The tangential polar equation of such an involute is

$$p=a\sin\phi\cos\phi+c,$$

therefore the radius of curvature of this at any point

$$=p+\frac{d^2p}{d\theta^2}=a\sin\phi\cos\phi+c-4a\sin\phi\cos\phi=c-3a\sin\phi\cos\phi,$$

and the curve will have no cusps unless c lie between $\frac{3}{2}a$ and $-\frac{3}{2}a$.

If $c=\frac{3}{2}a$, the equation of the line in any position is

$$x\cos\theta+y\sin\theta=\tfrac{3}{2}a+a\sin\theta\cos\theta,$$

and for its point of contact with its envelope,

$$-x\sin\theta+y\cos\theta=a\cos 2\theta,$$

$$\therefore\quad x=\tfrac{3}{2}a\cos\theta+a\sin\theta\cos^2\theta-a\sin\theta\cos 2\theta=\tfrac{3}{2}a\cos\theta+a\sin^3\theta,$$

$$y=\tfrac{3}{2}a\sin\theta+a\sin^2\theta\cos\theta+a\cos\theta\cos 2\theta=\tfrac{3}{2}a\sin\theta+a\cos^3\theta;$$

$$\therefore\quad x-y=a(\cos\theta-\sin\theta)(\tfrac{3}{2}-\cos^2\theta-\sin\theta\cos\theta-\sin^2\theta)=\tfrac{1}{2}a(\cos\theta-\sin\theta)^3,$$

$$x+y=a(\cos\theta+\sin\theta)(\tfrac{3}{2}+\cos^2\theta-\sin\theta\cos\theta+\sin^2\theta)$$

$$=\tfrac{1}{2}a(\cos\theta+\sin\theta)\left\{4+(\cos\theta-\sin\theta)^2\right\};$$

$$\therefore\quad x+y-2a\sqrt{2}=2a(\cos\theta+\sin\theta-\sqrt{2})+\tfrac{1}{2}a(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)^2$$

$$=2a\sqrt{2}\left\{\cos(\theta-\tfrac{1}{4}\pi)-1\right\}+a\sqrt{2}\cos(\theta-\tfrac{1}{4}\pi)\sin^2(\theta-\tfrac{1}{4}\pi);$$

hence, when $\theta=\frac{1}{4}\pi+\phi$, and ϕ is small,

$$x-y=a\sqrt{2}\sin^3\phi=a\sqrt{2}\,\phi^3,$$

$$x+y-2a\sqrt{2}=a\sqrt{2}\cos\phi\sin^2\phi-4a\sqrt{2}\sin^2\tfrac{1}{2}\phi$$

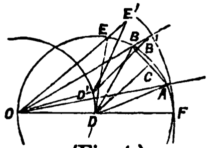
$$=4a\sqrt{2}\sin^2\tfrac{1}{2}\phi(\cos\phi\cos^2\tfrac{1}{2}\phi-1)=-4a\sqrt{2}\sin^4\tfrac{1}{2}\phi(1+2\cos^2\tfrac{1}{2}\phi)$$

$$=-\frac{3a}{2\sqrt{2}}\phi^4, \text{ approximately.}$$

Hence the singular points of this curve are not cusps, but are of the nature of a conjugate point lying on the curve, since $(x-y)^4 \propto (2a\sqrt{2}-x-y)^3$. This whole curve is rather like an ellipse of axes $4a$, $2a$, inclined at 45° and 135° to OA, but its area is less than that of the ellipse, and it lies altogether within it, touching it at the ends of the axes. Its area $= \frac{15}{8}\pi a^2$, and its length is $3\pi a$. The equation of the curve referred to its axes of asymptotes is $$(x^2+y^2-4a^2)^3+27a^2y^4 = 0.$$

II. *Solution by the* Rev. John George Birch, M.A.

1. Let OA and OB be the lines fixed in space, along which move the two points A and B, fixed in the lamina. Bisect AB in C, and draw AD, making the angle DAC equal to the complement of the angle AOB, and meeting the perpendicular to AB at its middle point in D. Then, as ADB is double of AOB, a circle with centre at D passes through the points O, A, B. The radius of this circle drawn on the lamina is a fixed magnitude, as it depends wholly on two fixed magnitudes AB and the angle AOB.



(Fig. 4.)

Any point E on this circle describes a right line in space passing through O. This may be proved thus. Let B′ and D′ be new positions of B and D; let the angle B′D′E′ be made equal to BDE; then D′E′ is equal to DE. For the inclination between the lines BD and B′D′ being equal to the inclination between the lines DE and D′E′, the difference between the angles OBD and OB′D′ is equal to the difference between OED and OE′D′. But the difference between the first pair of angles is, on account of BD and B′D′ being equal to OD and OD′ respectively, equal to DOD′. DOD′ is then equal to the difference between OED and OE′D′; that is, to the difference between EOD and OE′D′. Hence the angle D′OE′ is equal to OE′D′, and OD′ to D′E′. Therefore any point E of the circle OEA fixed in the lamina describes a right line in space passing through O.

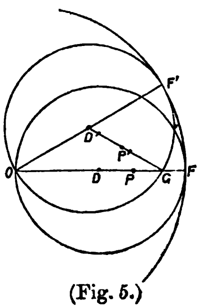
Now the motion of a circle, whose points describe right lines in space radiating from a fixed point, may be represented by making it roll inside another circle double its radius. Hence the motion of the lamina invariably attached to the circle OBAF may be represented by making this circle roll on another, BF, of double its radius fixed in space.

2. Each point of the lamina describes an ellipse.

Let OF be the circle fixed in the lamina whose points describe right lines, FF′ the circle of double the radius fixed in space on which it rolls, and P any point in the lamina. Take for axis of x the radius in which OD and DP are in the same line, and P at its greatest distances from O, and let OF′ be a new position of the circle OF. Join D′, the centre of this new circle, to the point of intersection G, and make D′P′ equal to DP. The locus of P′ is sought. Make $OD=r$, $DP=p$, $\angle DOD' = \alpha$, and let x and y be coordinates of P′;

then $\quad x=r\cos\alpha+p\cos\alpha, \quad y=r\sin\alpha-p\sin\alpha$;

and locus of P′ is the ellipse
$$\frac{x^2}{(r+p)^2}+\frac{y^2}{(r-p)^2} = 1,$$



(Fig. 5.)

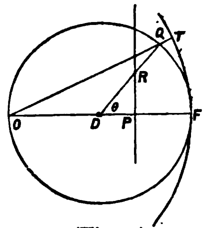
the distance of its focus from O being equal to $2\,(rp)^{\frac{1}{2}}$. Hence all the points on the lamina, situated on any radius through D, describe a series of ellipses in space, the direction of whose axes coincide; while all the points situated on any circle described around D as centre describe ellipses which differ only in the direction of their axes.

3. The foci of the ellipses described by points situated on a right line in the lamina lie on a rectangular hyperbola.

Let OF denote the same circle as before, D the same point, and RP the line fixed in the lamina, and FT the circle fixed in space. Take for axis of x the radius perpendicular to RP. Produce DR to meet the circle OF in Q; join OQ, and produce it to T. Make OD $= r$, DP $= p$, and RDP $= \theta$. Then the focus of the ellipse described by R lies on OT at a distance $2\,(rp\sec\theta)^{\frac{1}{2}}$, and x and y, its coordinates, are

$$x = 2\,(rp\sec\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}\theta, \qquad y = 2\,(rp\sec\theta)^{\frac{1}{2}}\sin\tfrac{1}{2}\theta.$$

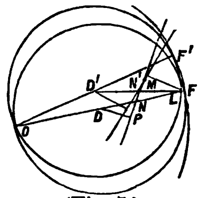
Eliminating θ, we have, for the locus of the foci of the ellipses described by the points of the right line RP, the rectangular hyperbola $x^2 - y^2 = 4rp$.



(Fig. 6.)

4. Every line in the lamina envelopes an involute of a four-cusped hypocycloid.

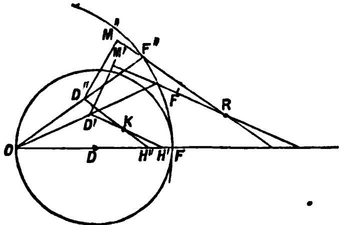
Let OF be the circle fixed in the lamina, MP the line on the lamina. Then MP touches its envelope at the foot of the perpendicular from F on it. Let the letters with a dash denote the corresponding points in a new position. Let L be the point where the new circle cuts OF. Join D′L. Then, as the angles NMN′ and NLN′ are equal, a circle can be described through MLNN′. In the limiting position this circle is described on NF as diameter. Therefore PMF is a right angle, and MP touches its envelope at the foot of the perpendicular from F.



(Fig. 7.)

As all lines parallel to MP touch their envelopes along the perpendicular MF, all these envelopes are the involutes of the same curve. It will therefore be sufficient to ascertain of what curve the envelope of the line parallel to MP through D is the involute.

This curve can be shown to be a four-cusped hypocycloid. Let OF be the circle fixed in the lamina whose points describe right lines, and let FF′F″ be the circle of double its radius fixed in space on which it rolls; and let D′ and D″ be two consecutive positions of the centre of the first circle, and H′ and H″ the points in which the consecutive circles, with D′ and D″ as centres, cut OF, and let K be the point in which D″H″ and D′H′ intersect. The line which was originally perpendicular to OF, through the point D, takes the two consecutive positions D′M′ and D″M″ respectively perpendicular to D′H′ and D″H″ through D′ and D″. And



(Fig. 8.)

these two lines touch the envelope at M′, M″, the feet of the perpendicular from F′ and F″ on them. Producing F′M′ and F″M″ to meet in R, R is a point on the curve whose involute is the envelope of M′D′, M″D″. It is readily seen that F′R is double D′K, and as the ultimate value of D′K is $r\cos 2\alpha$, the ultimate value of F′R is $2r\cos 2\alpha$ (where $\alpha = \angle$ DOD′), and the coordinates x and y of R are

$$x = 2r\cos\alpha\,(1+\cos 2\alpha) = 4r\cos^3$$
$$y = 2r\sin\alpha\,(1-\cos 2\alpha) = 4r\sin^3\alpha.$$

Hence the locus of R is the four-cusped hypocycloid

$$\left(\frac{x}{4r}\right)^{\frac{2}{3}} + \left(\frac{y}{4r}\right)^{\frac{2}{3}} = 1.$$

And all lines fixed in the lamina envelope the involutes of this curve.

4240. (Proposed by R. Tucker, M.A.)—Prove that the ellipses

$$a^2y^2 + b^2x^2 = a^2b^2, \quad a^2x^2\sec^4\phi + b^2y^2\operatorname{cosec}^4\phi = a^4e^4 \;\ldots\ldots\ldots\; (1, 2)$$

are so related that the envelope of (2), for different values of ϕ, is the evolute of (1); and that a point of contact of (2) with its envelope is the centre of curvature at the point of (1) whose eccentric angle is ϕ.

I. *Solution by the* Rev. Dr. Booth, F.R.S.

Let $U \equiv a^2y^2 + b^2x^2 - a^2b^2 = 0, \quad V \equiv a^2x^2\sec^4\phi + b^2y^2\operatorname{cosec}^4\phi - a^4b^4 = 0.$

Find the value of ϕ from the equation $\dfrac{dV}{d\phi} = 0$; substitute this value in $V = 0$, and we shall have $W \equiv (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} - (a^2-b^2)^{\frac{2}{3}} = 0$, and this is the projective equation of the evolute of $U = 0$.

Again, assuming

$$x = \frac{(a^2-b^2)\cos^3\phi}{a}, \text{ and } y = \frac{(a^2-b^2)}{b}\sin^3\phi \;\ldots\ldots\ldots\ldots\; (a),$$

we shall find that these values of x and y satisfy the equations $V = 0$, $W = 0$, hence this point is common to the ellipse $V = 0$ and its evolute $W = 0$.

Moreover, if $\bar{x}$ and $\bar{y}$ be the coordinates of a point on the ellipse $U = 0$, of which point ϕ is the eccentric angle; we shall have

$$\bar{x} = a\cos\phi, \text{ and } \bar{y} = b\sin\phi \;\ldots\ldots\ldots\ldots\; (b);$$

and if we eliminate $\cos\phi$ and $\sin\phi$ between (a) and (b), we shall have

$$\bar{x} - \frac{a^{\frac{4}{3}}x^{\frac{1}{3}}}{(a^2-b^2)^{\frac{1}{3}}}, \quad \bar{y} - \frac{b^{\frac{4}{3}}y^{\frac{1}{3}}}{(a^2-b^2}.$$

Hence x and y are the projective coordinates of the centre of curvature of the point $(\bar{x}, \bar{y},)$.

2. The question may be solved as follows by tangential coordinates:—

Let $U' \equiv a^2\xi^2 + b^2v^2 - 1 = 0$

and $V' \equiv \dfrac{(a^2-b^2)^2}{a^2}\cos^4\phi\,\xi^2 + \dfrac{(a^2-b^2)^2}{b^2}\sin^4\phi\,v^2 - 1 = 0$

be the tangential equations of the two ellipses. Then, finding the value of $\frac{dV'}{d\phi}=0$, we shall have $\cos^2\phi=\frac{a^2v^2}{a^2v^2+b^2\xi^2}$; eliminating $\sin\phi$, $\cos\phi$ from $V'=0$, we shall have $W'\equiv a^2v^2+b^2\xi^2-(a^2-b^2)^2\xi^2v^2=0$,
which is the tangential equation of the evolute of $U=0$. (See BOOTH's *New Geometrical Methods*, p. 115.)

Assume $\xi=\frac{a\sec\phi}{(a^2-b^2)}$, and $v=\frac{b\operatorname{cosec}\phi}{(a^2-b^2)}$ .... (c).

Now substituting these assumed values of ξ and v in the equations $V'=0$, and $W'=0$, we shall find that they satisfy those equations; consequently, the ellipse $V'=0$, and its evolute $W'=0$, have a common tangent.

Let $\bar{\xi}$ and ξ denote the tangential ordinates along the axis of X, made by two tangents passing through a point on an ellipse, one to the ellipse, the other to the evolute, and let ϕ be the excentric angle of $U'=0$ at this point; then

$$a\bar{\xi}=\cos\phi,\quad b\bar{v}=\sin\phi,\quad\text{and}\quad a\bar{\xi}\xi=\xi\cos\phi=\frac{a}{a^2-b^2},\ \text{from } (a).$$

Hence $(a^2-b^2)\bar{\xi}\xi=1$; consequently the common tangent to $V'=0$ and $W'=0$ passes through the point on $U'=0$, of which the excentric angle is ϕ.

If we substitute the values of x, y, ξ, v assumed in the equations (a) and (c), we shall find that they satisfy the *dual equation* $x\xi+yv=1$; consequently the common tangent passes through the common point of the two given ellipses.

II. *Solution by the* PROPOSER.

Let $\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$ be the concentric ellipse touching the evolute of (1), whose equation is $(a\alpha)^{\frac{2}{3}}+(b\beta)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.
Since the two touch, at their point of contact we have, since

$$\alpha=ae^2\cos^3\phi,\quad\text{and}\quad\beta=\frac{a^2e^2}{b}\sin^3\phi,$$

$$\tan^4\phi=\frac{B^2}{A^2}\cdot\frac{b^2}{a^2},\ \text{and}\ \frac{a^2e^4\cos^6\phi}{A^2}+\frac{a^4e^4\sin^6\phi}{B^2}=1\ \text{.........}\ (3,4);$$

whence $B^2b^2=a^4e^4\sin^4\phi$, $A^2a^2=a^4e^4\cos^4\phi$.
This gives (2), whose envelope is now easily seen to be evolute of (1).

From (2) and the corresponding equation (5) for the point conjugate to ϕ, we readily see that (2) and (5) intersect on the lines $y=\pm\frac{b}{a}x$.

III. *Solution by* E. B. ELLIOTT.

The normal to (1) at the point ϕ is

$$ax\sec\phi-by\operatorname{cosec}\phi=a^2e^2\ \text{........................}\ (3).$$

Differentiating, $ax\sec\phi\tan\phi+by\operatorname{cosec}\phi\cot\phi=0$ .... (4);
therefore, squaring (3) and (4) and adding the results, we have

$$a^2x^2\sec^4\phi+b^2y^2\operatorname{cosec}^4\phi=a^4e^4.$$

Thus the centre of curvature at any point of (1) lies on the corresponding ellipse of the series (2).

Again, by (3) and (4), the coordinates of this point are $ae^2\cos^3\phi$ and $-\frac{a^2e^2}{b}\sin^3\phi$; therefore (3) is the tangent at it to (2). Thus the envelope of (2) is also that of (3), that is, it is the evolute of (1).

4119. (Proposed by Professor CROFTON, F.R.S.)—1. If two points be taken at random within any convex boundary, of perimeter L and area Ω, the chance that a straight line drawn at random across the area shall pass between the two points is $p=\frac{1}{3L\Omega^2}\iint C^4 dp\,d\theta$, where C is the length of a variable chord drawn across the boundary, and p, θ are the perpendicular on C from any fixed pole, and its inclination to a fixed axis; the integration extending to all positions of C.

2. In the same case, if *two* random straight lines are drawn, the chance that *both* shall pass between the points, is $p=8k^2L^{-2}$, where k is the radius of gyration of the area Ω round its centre of gravity.

Solution by Colonel CLARKE, C.B., F.R.S.

1. We have here first to find the sum of the distances of all pairs of points within the convex area. Through any point A within the area draw two chords PQ, P′Q′, making angles θ, $\theta+d\theta$ with any fixed line. Let AQ $=r$, PQ $=$ C, then the sum of the distances of all the points B within the space AQQ′ from A is

$$\int_0^r r^2 dr\,d\theta = \tfrac{1}{3}r^3 d\theta,$$

and the sum of the distances from A of all points B that lie between PQ and P′Q′ is $\qquad \tfrac{1}{3}\left\{r^3+(C-r)^3\right\}d\theta.$

Now if A be anywhere within the element of area dA, the sum of the distances of all points A, B, whose relative azimuth, so to speak, is between θ and $\theta+d\theta$, is $\qquad \tfrac{1}{3}\left\{r^3+(C-r)^3\right\}dA\,d\theta.$

Next, let A take any position between the chord PQ and another chord parallel to it at the distance dp, then $dA=dr\,dp$, and the sum of the distances of such pairs of points

$$=\tfrac{1}{3}d\theta\,dp\int_0^C\left\{r^3+(C-r)^3\right\}dr=\tfrac{1}{6}d\theta\,dp\,C^4.$$

Integrating next with respect to p, we get the sum of the distances of all pairs of points of which the relative azimuth is between θ and $\theta+d\theta$; and integrating again with respect to θ, we get the sum of the distances of *all* pairs of points $\qquad =\frac{1}{6}\iint C^4 dp\,d\theta.$

Hence, since the number of lines crossing a finite line of length $=a$ is $2a$; and since the total number of lines and pairs of points is $L\Omega^2$, the required probability is $$p = \frac{1}{3L\Omega^2}\iint C^4 dp\, d\theta.$$

2. The second part of this question is evident from the solution to Question 4093 (*Educational Times* for Sept., or *Reprint*, Vol. XX., p. 33).

[Prof. CROFTON remarks that the expression for p in part (1) of this Question can easily be applied to deduce Mr. WATSON's Solution for the case of a triangle; *Reprint*, Vol. XVIII., p. 47.]

4153. (Proposed by A. MARTIN.)—If m, m_1, m_2, m_3 be the several distances of the centre of the nine-point circle from the centres of the inscribed and escribed circles of a triangle, prove that
$$m + m_1 + m_2 + m_3 = 6R.$$

Solution by CHRISTINE LADD.

The radius of the nine-point circle is $\frac{1}{2}R$, and this circle touches the inscribed and escribed circles; hence we have
$$m = \tfrac{1}{2}R - r, \quad m_1 = \tfrac{1}{2}R + r_1, \quad m_2 = \tfrac{1}{2}R + r_2, \quad m_3 = \tfrac{1}{2}R + r_3;$$
therefore $$\Sigma(m) = 2R + r_1 + r_2 + r_3 - r = 6R.$$

4191. (Proposed by G. O'HANLON.)—Prove that the oscillations of all particles of a sphere, if allowed to move through straight smooth tubes infinitely small in bore, are isochronous.

Solution by the PROPOSER.

Draw a plane through the centre of the sphere, and perpendicular to the tube. Then, since the attraction on the particle varies as its distance from the centre of the sphere, on being resolved along the tube it will vary as the distance of the particle from the plane. But when this is the case, the oscillations are isochronous.

4237. (Proposed by Professor CROFTON, F.R.S.)—A straight line drawn at random crosses any convex area; let p_1 be the probability that it passes between two points taken at random in the area; again, let p_2 be the probability that it meets the triangle formed by three points taken at random in the area; then prove that $p_2 = \frac{3}{2}p_1$.

I. *Solution by* E. B. ELLIOTT.

The line must meet two of the sides in order to cut the triangle.
Now the chance that it will cut the triangle by meeting two of the sides, of which the first is one, is p_1; and the chance that it will meet the second and third is $\frac{1}{2}p_1$, since it is equally likely to meet the second and third as the second and first. Thus adding, $p_2 = p_1 + \frac{1}{2}p_1 = \frac{3}{2}p_1$.

II. *Solution by* Colonel CLARKE, C.B., F.R.S.

Let $z = f(xy)$ be the sum of the distances of all the points within the convex area from the point xy, then the sum of the distances of all pairs of points A, B will be $\int z\,d\Omega$, the integration being carried over the whole area Ω; hence the chance of a random line which crosses the contour passing between a pair of random points is, L being the perimeter,

$$p_1 = \frac{2}{L\Omega^2}\int z\,d\Omega.$$

Now let A, B, C be three points within the area, the number of lines which cross this triangle is $a+b+c$, and the sum of the perimeters of all the triangles is
$$U = \iiint (a+b+c)\,dA\,dB\,dC,$$
where dA, dB, dC are elements of area at A, B, C. Let, first, C vary, A and B being fixed,
$$U = \iint (z_a + z_b + c\Omega)\,dA\,dB.$$

Then, making B variable, we have

$$U = \int \left(z_a\Omega + \int z\,d\Omega + z_a\Omega\right) dA, \text{ which, finally, } = 3\Omega\int z\,d\Omega.$$

The total number of lines and triads of points being $L\Omega^3$, we have

$$p_2 = \frac{3}{L\Omega^2}\int z\,d\Omega; \text{ hence } p_2 = \tfrac{3}{2}p_1.$$

4102. (Proposed by T. T. WILKINSON, F.R.A.S.)—Prove the following Rule for finding right-angled triangles whose legs differ by a given number: —Write the sides of the original triangle under each other with the hypotenuse second. Then make a second column thus: take the sum of the two upper figures for a new *top* figure; the sum of the two lower for a new *bottom* figure; and the sum of all three for a new *middle* figure. Repeat the operation on this second column so as to get a third, and this column will represent the sides of a new triangle fulfilling the conditions. Continuing the process to any number of columns, the alternate ones, namely, each that contains *two odd* numbers, will give a prime right-angled triangle with the same

difference of legs as the first. The following are several such triangles:—

	I		II		III		IV	V	VI	
Base	3	8	20	49	119	288	696	4059	23660	
Hyp.	5	12	29	70	169	408	985	5741	33461	&c.
Perp.	4	9	21	50	120	289	697	4060	23661	

Solution by Professor WOLSTENHOLME, M.A.

The sides of any right-angled triangle may be taken to be p^2-q^2, $2pq$, p^2+q^2; and if the two legs differ by 1, we must have either $p^2-q^2-2pq=1$ or $2pq+q^2-p^2=1$; *i.e.*, $(p-q)^2-2q^2=1$ or $(q+p)^2-2p^2=1$. Now the numbers which satisfy the equation $x^2-2y^2=1$ are the numerators and denominators of the odd terms of the convergents to $\sqrt{2}$, viz. $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, ..., and both numerators and denominators are formed according to the rule $u_n=2u_{n-1}+u_{n-2}$; the nth numerator being $\dfrac{(1+\sqrt{2})^{n+1}+(1-\sqrt{2})^{n+1}}{2}$, and the nth denominator $\dfrac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n-1}}{2\sqrt{2}}$. We shall get the same values for the sides, whichever system we use; taking the first, we have, the convergent being odd,

$$p-q=\frac{(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}}{2},\text{ and } q=\frac{(1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n}}{2\sqrt{2}};$$

$$\therefore\ p=\frac{(1+\sqrt{2})^{2n+1}-(1-\sqrt{2})^{2n+1}}{2\sqrt{2}},\quad p+q=\frac{(1+\sqrt{2})^{2n+1}+(1-\sqrt{2})^{2n+1}}{2}.$$

Hence
$$p^2-q^2=\frac{(1+\sqrt{2})^{4n+1}+(1-\sqrt{2})^{4n+1}+2}{4},$$

$$2pq=\frac{(1+\sqrt{2})^{4n+1}+(1-\sqrt{2})^{4n+1}-2}{4},$$

$$p^2+q^2=\frac{(1+\sqrt{2})^{4n+1}-(1-\sqrt{2})^{4n+1}}{2\sqrt{2}},$$

the general form for the sides of a right-angled triangle in which the legs differ by 1. The rule given is proved at once by writing down

$$\begin{array}{c|c|c} p^2-q^2 & 2p^2 & 4p^2+2pq \\ p^2+q^2 & 2p^2+2pq & 5p^2+4pq+q^2 \\ 2pq & (p+q)^2 & 3p^2+4pq+q^2 \end{array}$$

in the third column the difference of the top and bottom term is $p^2-2pq-q^2$, the same as in the first column; also the square of the middle term is the sum of the squares of the other two. Also if the numbers in the first column be prime to each other, so also must those in the third, since they are only formed by successive additions and subtractions.

The numbers given may be found at once from the convergents to $\sqrt{2}$, taking the *even* ones, by taking the denominator for the hypotenuse, and two numbers differing by 1, and whose sum is the numerator, for the legs. The convergents are $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, $\frac{41}{29}$, $\frac{99}{70}$, $\frac{239}{169}$, $\frac{577}{408}$, $\frac{1393}{985}$, whence the triangles

$$\begin{array}{c|c|c|c|} 3 & 20 & 119 & 696 \\ 5 & 29 & 169 & 985 \\ 4 & 21 & 120 & 697 \end{array}$$

......, and the rule might be modified so as to give each directly from the preceding as follows: to each leg add the whole

perimeter and the hypotenuse; and to the hypotenuse add twice the perimeter. The relation between any three successive odd or three successive even numerators or denominators is of course $u_n = 6u_{n-1} - u_{n-2}$, which rule would suffice for the hypotenuse; the rule for the shorter leg being $u_n = 6u_{n-1} - u_{n-2} + 2$, and for the longer $u_n = 6u_{n-1} - u_{n-2} - 2$. The nth hypotenuse is the coefficient of x^{n-1} in the expansion of $\dfrac{5-x}{1-6x+x^2}$, the legs of the nth triangle are the coefficients of x^{n-1} in

$$\frac{1}{2}\left(\frac{7-x}{1-6x-x^2} \mp \frac{1}{1-x}\right).$$

PYTHAGOREAN TRIANGLES. *By* S. TEBAY, M.A.

The following proof of Mr. Wilkinson's rule* shows that no other relation exists producing Pythagorean triangles of given side-differences.

If a, b, c be the sides of a right-angled triangle, and we form the following columns:—

$$\begin{aligned} a' &= a + c, & \mathrm{A} &= a' + c' = 2a + b + 2c, \\ b' &= b + c, & \mathrm{B} &= b' + c' = a + 2b + 2c, \\ c' &= a + b + c, & \mathrm{C} &= a' + b' + c' = 2a + 2b + 3c; \end{aligned}$$

then A, B, C are the sides of a right-angled triangle, such that $\mathrm{A} - \mathrm{B} = a - b$. Assume

$$\mathrm{A} = la + mb + nc, \quad \mathrm{B} = l'a + m'b + n'c, \quad \mathrm{C} = l''a + m''b + n''c.$$

The condition $\mathrm{A} - \mathrm{B} = a - b$ gives $l - l' = 1$, $m - m' = -1$, $n - n' = 0$.

Now
$$\begin{aligned} \mathrm{A}^2 &= l^2a^2 + m^2b^2 + n^2c^2 + 2lmab + 2lnac + 2mnbc, \\ \mathrm{B}^2 &= l'^2a^2 + m'^2b^2 + n'^2c^2 + 2l'm'ab + 2l'n'ac + 2m'n'bc \\ &= (l-1)^2a^2 + (m+1)^2b^2 + n^2c^2 + 2(l-1)(m+1)ab \\ &\quad + 2(l-1)nac + 2(m+1)nbc; \end{aligned}$$

therefore
$$\begin{aligned} \mathrm{A}^2 + \mathrm{B}^2 = 2l(l-1)a^2 + 2m(m+1)b^2 + (2n^2+1)c^2 + 2(lm + l - m - 1)ab \\ + 2n(2l-1)ac + 2n(2m+1)bc. \end{aligned}$$

Comparing this with

$$\mathrm{C}^2 = l''^2a^2 + m''^2b^2 + n''^2c^2 + 2l''m''ab + 2l''n''ac + 2m''n''bc,$$

we have
$$\begin{aligned} l''^2 &= 2l(l-1), & l''m'' &= 2lm + l - m - 1, \\ m''^2 &= 2m(m+1), & l''n'' &= n(2l-1), \\ n''^2 &= 2n^2 + 1, & m''n'' &= n(2m+1). \end{aligned}$$

These equations give

$$l''^2 = \frac{(2lm + l - m - 1)(2l-1)}{2m+1} = 2l(l-1),$$

$$m''^2 = \frac{(2lm + l - m - 1)(2m+1)}{2l-1} = 2m(m+1),$$

$$n''^2 = \frac{(2l-1)(2m+1)n^2}{2lm + l - m - 1} = 2n^2 + 1;$$

from which we find $l - m = 1$, $n^2 = 2lm$.

* Given in the foregoing Question 4102.

Therefore $$l = \tfrac{1}{2}\left\{\sqrt{(2n^2+1)}+1\right\}.$$

Let $$2n^2+1 = \left(\frac{p}{q}n \pm 1\right)^2; \text{ therefore } n = \frac{\pm 2pq}{p^2-2q^2}.$$

The condition $p^2-2q^2 = -1$ is satisfied by taking $p=1$, $q=1$; therefore
$$n = l = m' = n' = l'' = m'' = 2, \quad m = l' = 1, \quad n'' = 3;$$
therefore $\quad \mathrm{A} = 2a+b+2c, \quad \mathrm{B} = a+2b+2c, \quad \mathrm{C} = 2a+2b+3c.$

If x be any integer, we can take
$$p = \tfrac{1}{2}\left\{(1+\sqrt{2}) + (1-\sqrt{2})\right\} = \tfrac{1}{2}\left\{\eta^x + (-1)^x\eta^{-x}\right\},$$
$$q = \frac{1}{2\sqrt{2}}\left\{(1+\sqrt{2})^x - (1-\sqrt{2})^x\right\} = \frac{1}{2\sqrt{2}}\left\{\eta^x - (-1)^x\eta^{-x}\right\};$$
therefore $$n = \pm\frac{1}{2\sqrt{2}}\left\{\eta^{2x} - \eta^{-2x}\right\}, \qquad l = \tfrac{1}{4}\left\{\eta^x + (-1)^x\eta^{-x}\right\}^2,$$
$$n'' = 2l - 1 = \tfrac{1}{2}\left\{\eta^{2x} + \eta^{-2x}\right\}, \text{ \&c.};$$
therefore
$$a_x = \tfrac{1}{4}(a+b\pm c\sqrt{2})\eta^{2x} + \tfrac{1}{4}(a+b\mp c\sqrt{2})\eta^{-2x} + \tfrac{1}{2}(a-b),$$
$$b_x = \tfrac{1}{4}(a+b\pm c\sqrt{2})\eta^{2x} + \tfrac{1}{4}(a+b\mp c\sqrt{2})\eta^{-2x} - \tfrac{1}{2}(a-b),$$
$$c_x = \pm\frac{1}{2\sqrt{2}}(a+b\pm c\sqrt{2})\eta^{2x} \pm \frac{1}{2\sqrt{2}}(a+b\mp c\sqrt{2})\eta^{-2x}.$$

Thus we immediately deduce
$$a_{x\pm 1} = 2a_x + b_x \pm 2c_x, \quad b_{x\pm 1} = a_x + 2b_x \pm 2c_x, \quad c_{x\pm 1} = \pm 2a_x \pm 2b_x + 3c_x.$$

To satisfy these relations, a, b, c need not necessarily be integers; they may be whole or fractional, positive or negative, irrational or unreal, provided $a^2+b^2=c^2$. If a, b, c be positive integers, for instance, the prime sides of any series of triangles produced by this rule, these will be prime to one another; and from any one of them, by reversing the process, the others can be successively reproduced, till we arrive at the prime a, b, c; after which, one side will become negative, and by changing the sign, and repeating the operation, we at last arrive at the triangle 3, 4, 5, (not $3m$, $4m$, $5m$) as pointed out by Mr. Monck, unless the factor m has been previously introduced.

The following are remarkable relations :—
$$a_{x+3} - 7a_{x+2} + 7a_{x+1} - a_x = 0, \qquad b_{x+3} - 7b_{x+2} + 7b_{x+1} - b_x = 0,$$
$$c_{x+3} - 7c_{x+2} + 7c_{x+1} - c_x = 0.$$

These are, in fact, the same as
$$a_{x+2} - 6a_{x+1} + a_x + 2m = 0, \; b_{x+2} - 6b_{x+1} + b_x - 2m = 0, \; c_{x+2} - 6c_{x+1} + c_x = 0.$$

Note on Pythagorean Triangles (Question 4102.)

By T. T. Wilkinson, F.R.A.S.

If we take the sides of the Pythagorean triangle as x^2-y^2, x^2+y^2, and $2xy$, we shall have the columns as follows :—

I.	(a.)	II.	(b.)	III.
x^2-y^2	$2x^2$	$4x^2+2xy$	$9^2x+6xy+y^2$	$21x^2+16xy+3y^2$
x^2+y^2	$2x^2+2xy$	$5x^2+4xy+y^2$	$12x^2+10xy+2y^2$	$29x^2+24xy+5y^2$
$2xy$	$x^2+2xy+y^2$	$3x^2+4xy+y^2$	$8x^2+8xy+2y^2$	$20x^2+18xy+4y^2$

&c. &c.

But the expressions in (a) are obviously composed of the following quantities:—$(x^2+y^2)+(x^2-y^2)$, $(x^2+y^2)+(x^2-y^2)+2xy$, $(x^2+y^2)+2xy$; and those in II. are easily seen to be $2(x^2+y^2)+2(x^2-y^2)+2xy$, $3(x^2+y^2)+2(x^2-y^2)+4xy$, and $2(x^2+y^2)+(x^2-y^2)+4xy$. Whence the rule given in Question 4102 follows at once.

If n = the difference of the sides, $n=1$ gives the series (3, 5, 4), (8, 12, 9), (20, 29, 21), (49, 70, 50), &c.; where the *odd* groups give Pythagorean triangles, and the *even* groups give $8^2+9^2=12^2+1^2$, $49^2+50^2=70^2+1^2$; and generally, for any given difference n, the even columns give base^2 + perp.^{2} = hyp.$^2+n^2$. In the above expressions, when $x=1$, $y=0$, the sides are 1, 1, 0; which may be termed the fundamental column, although no real triangle is formed. The first real triangle for difference $n=1$, is 3, 4, 5; for $n=2$, it is 6, 10, 8; and generally, for difference n, the first real triangle is $3n$, $4n$, $5n$.

If it were required to find a series of right-angled triangles, such that they shall have any given base, and the difference of the hypothenuse and perpendicular any given number, put x = hyp., $x-n$ = perp., and a = given base. Then $x^2-a^2=(x-n)^2=x^2-2nx+n^2$; whence $x=\dfrac{a^2+n^2}{2n}$; and $x-n=\dfrac{a^2-n^2}{2n}$. Now, in order that the sides may be rational, we must have a = some multiple of n, and also a and n both *even* or both *odd*.

Take $n=1$, $a=1$, then $x=1$, $x-1=0$, and the sides are 1, 1, 0; the former fundamental triangle. If $n=1$, $a=3$, then $x=5$, $x-n=4$; and the sides of the first real triangle are 3, 4, 5. Hence, if a = 1, 3, 5, 7, 9, &c., and $n=1$, we have the series

$$\left.\begin{matrix} 1, & 0, & 1, \\ 3, & 4, & 5, \\ 5, & 12, & 13, \\ 7, & 24, & 25, \\ 9, & 40, & 41, \text{ \&c.} \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But if a = 2, 4, 6, 8, 10, 12, &c., and $n=2$, we have the series

$$\left.\begin{matrix} 2, & 0, & 2, \\ 4, & 3, & 5, \\ 6, & 8, & 10, \\ 8, & 15, & 17, \\ 10, & 24, & 26, \text{ \&c.} \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

When a = 2, 4, 6, 8, &c., and $n=1$, we have the series

$$\left.\begin{matrix} 2, & 1{\cdot}5, & 2{\cdot}5, \\ 4, & 7{\cdot}5, & 8{\cdot}5, \\ 6, & 17{\cdot}5, & 18{\cdot}5, \text{ \&c.,} \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

which obviously follows from the preceding group (2) by dividing the *odd* rows by $n=2$.

4174. (Proposed by the Editor.)—If α, β, γ be the radii of three circles that have a common tangent; a, b, c, A, B, C, Δ, p_1, p_2, p_3 the sides, angles, area, and perpendiculars from the vertices on the opposite sides, of the triangle connecting the centres of the circles; prove that

$$\pm\left\{a^2-(\beta\pm\gamma)^2\right\}^{\frac{1}{2}}\pm\left\{b^2-(\gamma\pm\alpha)^2\right\}^{\frac{1}{2}}\pm\left\{c^2-(\alpha\pm\beta)^2\right\}^{\frac{1}{2}}=0,$$

$$a^2(\alpha-\beta)(\alpha-\gamma)+b^2(\beta-\gamma)(\beta-\alpha)+c^2(\gamma-\alpha)(\gamma-\beta)=4\Delta^2,$$

$$a^2\alpha^2+b^2\beta^2+c^2\gamma^2-2bc\beta\gamma\cos A-2ca\gamma\alpha\cos B-2ab\alpha\beta\cos C=4\Delta^2,$$

$$\frac{\alpha^2}{p_1^{\,2}}+\frac{\beta^2}{p_2^{\,2}}+\frac{\gamma^2}{p_3^{\,2}}-2\frac{\beta\gamma}{p_2p_3}\cos A-2\frac{\gamma\alpha}{p_3p_1}\cos B-2\frac{\alpha\beta}{p_1p_2}\cos C=1.$$

I. *Solution* (1) *by* J. C. Malet, M.A.; (2) *by the* Proposer.

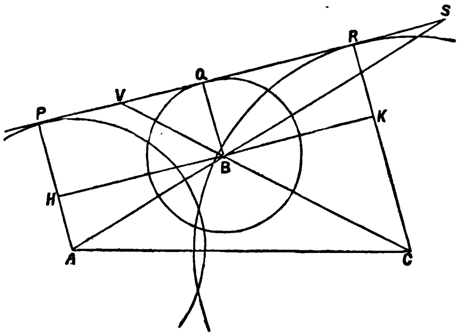
1. The lengths of the common tangents to the three pairs of circles are

$$\left\{c^2-(\alpha\pm\beta)^2\right\}^{\frac{1}{2}},\quad\left\{b^2-(\gamma\pm\alpha)^2\right\}^{\frac{1}{2}},\quad\left\{a^2-(\beta\pm\gamma)^2\right\}^{\frac{1}{2}};$$

hence, equating the sum of two of these expressions to the third, we see that the condition that a common tangent can be drawn to all three circles may be expressed in the *first* of the forms given in the Question, and, when rationalized, this would readily lead to the other three forms.

Certain limitations with regard to the ambiguous signs are necessary. The sign of one of the radicals must always be different from the signs of the other two. With regard to the ambiguous signs inside the radicals, the only limitation necessary is that they can never be all positive.

2. Let P, Q, R be the points of contact of the common tangent PQR; and let S, V be the points at which the lines of centres AB, CB meet the common tangent; also, let HBK be drawn parallel to the line of centres AC, and therefore perpendicular to the radii AP, BQ, CR.



Then, if D be the diameter of the circle drawn round the triangle VBS, we have

$$VS=D\,.\sin B,\quad SB=D\,.\sin V=D\,.\sin CBK=D\left(\frac{\gamma-\beta}{a}\right),$$

$$BV=D\,.\sin S=D\,.\sin ABH=D\left(\frac{\alpha-\beta}{c}\right);$$

and $$SB^2+BV^2-2SB\,.\,BV\cos B=VS^2;$$

therefore $$\left(\frac{\gamma-\beta}{a}\right)^2+\left(\frac{\alpha-\beta}{c}\right)^2-2\left(\frac{\gamma-\beta}{a}\right)\left(\frac{\alpha-\beta}{c}\right)\cos B=\sin^2 B.$$

Multiplying by a^2c^2, putting $a^2-b^2+c^2$ for $\cos B$, and 2Δ for $ac\sin B$, the equation becomes

$$a^2(\alpha-\beta)^2+c^2(\gamma-\beta)^2-(\alpha-\beta)(\gamma-\beta)(a^2-b^2+c^2)=4\Delta^2,$$

which may be at once put into the *second* of the forms given in the Question. Developing the left hand side of this second equation, and putting $2bc\cos A$ for $-a^2+b^2+c^2$, &c., we obtain the *third* equation; and dividing both sides of this equation by $4\Delta^2 = a^2p_1^2 = b^2p_2^2 = c^2p_3^2$, we obtain the *fourth* equation.

II. *Solution by* Professor WOLSTENHOLME, M.A.

Let forces P, Q, R, acting along the sides BC, CA, AB of a triangle taken in order, have a resultant X; and let α, β, γ be the perpendiculars from A, B, C on the line of action of the resultant.

Since the moment of the resultant about any point is equal to the sum of the moments of the forces, by taking moments about A, B, C we have

$$X\alpha = Pp_1, \quad X\beta = Qp_2, \quad X\gamma = Rp_3;$$

also $$X^2 = P^2+Q^2+R^2-2QR\cos\alpha-2RP\cos\beta-2PQ\cos\gamma;$$

whence $$1 = \frac{\alpha^2}{p_1^2}+\frac{\beta^2}{p_2^2}+\frac{\gamma^2}{p_3^2}-\frac{2\beta\gamma}{p_2p_3}\cos A-\frac{2\gamma\alpha}{p_3p_1}\cos B-\frac{2\alpha\beta}{p_1p_2}\cos C.$$

Also, by taking moments about any point whose areal coordinates measured on A, B, C are x, y, z, we have

$$Pp_1x+Qp_2y+Rp_3z = X.\varpi, \quad \text{or} \quad \varpi = \alpha x+\beta y+\gamma z,$$

ϖ being the perpendicular from the point (x, y, z) on the straight line; the rule of signs of course being attended to in $\varpi, \alpha, \beta, \gamma$.

It follows that the perpendicular from any point (x, y, z) on the straight line $lx+my+nz = 0$, is

$$\frac{lx+my+nz}{\left(\frac{l^2}{p_1^2}+\frac{m^2}{p_2^2}+\frac{n^2}{p_3^2}-\frac{2mn}{p_2p_3}\cos A-\frac{2nl}{p_3p_1}\cos B-\frac{2lm}{p_1p_2}\cos C.\right)^{\frac{1}{2}}}$$

III. *Solution by* OMEGA.

Let the circles, whose radii are α, β, γ, and whose centres are at the angular points of the triangle of reference, touch the straight line whose equation in trilinear coordinates is

$$l\alpha+m\beta+n\gamma = 0.$$

Then, with Mr. WHITWORTH's notation (*Modern Geom.*, Art. 46) we have

$$\alpha = \frac{2l\Delta}{a\{l, m, n\}}, \quad \beta = \frac{2m\Delta}{b\{l, m, n\}}, \quad \gamma = \frac{2n\Delta}{c\{l, m, n\}};$$

or $$\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{c\gamma}{n} = \frac{2\Delta}{\{l, m, n\}};$$

whence $$\{a\alpha, b\beta, c\gamma\} = 2\Delta,$$

or $$a^2\alpha^2+b^2\beta^2+c^2\gamma^2-2bc\,\beta\gamma\cos A-2ca\,\gamma\alpha\cos B-2ab\,\alpha\beta\cos C = 4\Delta^2 \ldots (\beta),$$

or $$a^2(\alpha-\beta)(\alpha-\gamma)+b^2(\beta-\gamma)(\beta-\alpha)+c^2(\gamma-\alpha)(\gamma-\beta) = (a+b+c)^2r^2,$$

where r denotes radius of the circle inscribed in the triangle formed by joining the centres of the three circles.

[Mr. BILLS gives a solution which may be abridged as follows:—

Let $\quad PQ^2 = l,\ QR^2 = m,\ PR^2 = n;$

then $\quad l = c^2 - (\alpha-\beta)^2,\quad m = a^2 - (\gamma-\beta)^2,\quad n = b^2 - (\gamma-\alpha)^2;$

and $\quad \sqrt{l} + \sqrt{m} - \sqrt{n} = 0,\quad$ or $\quad l^2 + m^2 + n^2 - 2mn - 2nl - 2lm = 0.$

This is one form of the required relation: and it may readily be transformed to the other equations given in the Question.]

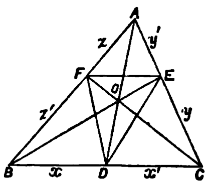
4184. (Proposed by the EDITOR.)—A triangle DEF has one of its vertices on each of the sides of a triangle ABC; prove (1) that

$$AF \,.\, BD \,.\, CE + AE \,.\, BF \,.\, CD = 4R \,.\, \Delta DEF,$$

where R is the radius of the circle ABC; and (2), that if the straight lines AD, BE, CF pass through the same point O, then

$$AF \,.\, BD \,.\, CE = AE \,.\, BF \,.\, CD = 2R \,.\, \Delta DEF.$$

I. *Solution by* H. S. MONCK; R. TUCKER, M.A.; *and others.*



1. Let each side be divided into two segments, x, x'; y, y'; z, z'; then

$$\Delta AFE : \Delta ABC = zy' : bc = azy' : abc,$$

with similar equations for ΔADF and ΔCED; therefore

$$\Delta DEF : \Delta ABC = abc - azy' - bxz' - cyx' : abc.$$

Substituting in the third member of this equation $x + x'$ for a, $y + y'$ for b, and $z + z'$ for c, it easily becomes $\quad \Delta DEF : \Delta ABC = xyz + x'y'z' : abc.$

But $abc = 4R \,.\, \Delta ABC$; therefore $xyz + x'y'z' = 4R \,.\, \Delta DEF.$

2. The preceding property is true for *any* inscribed triangle. But if the lines joining the vertices of the inscribed triangle to those of the triangle ABC pass through the same point O, it is known that $xyz = x'y'z'$; hence in this case we have $xyz = 2R \,.\, \Delta DEF.$

II. *Solution by* R. W. GENESE, B.A.; *the* Rev. J. L. KITCHIN, M.A.; *and others.*

When the lines pass through the same point O, let $BD = xa$, $CE = yb$, $AF = zc$; then the triangular coordinates of D, E, F are respectively

$$(0,\ 1-x,\ x),\ (y,\ 0,\ 1-y),\ (1-z,\ z,\ 0);$$

therefore area of triangle DEF $= \begin{vmatrix} 0, & 1-x, & x, \\ y, & 0, & 1-y, \\ 1-z, & z, & 0, \end{vmatrix} \Delta ABC$

$$= \left\{(1-x)(1-y)(1-z) + xyz\right\} \Delta ABC.$$

But by generating $\quad BD \,.\, CE \,.\, AF = DC \,.\, EA \,.\, FB;$

therefore $\quad ax \,.\, by \,.\, cz = a(1-x) \,.\, b(1-y) \,.\, c(1-z).$

Thus we find the area of the triangle DEF $= 2xyz \triangle \mathrm{ABC}$;

therefore $\mathrm{BD}\,.\,\mathrm{CE}\,.\,\mathrm{AF} = \mathrm{DC}\,.\,\mathrm{EA}\,.\,\mathrm{FB} = \dfrac{\text{area of DEF} \times abc}{2\triangle \mathrm{ABC}} = 2\mathrm{R}\triangle \mathrm{DEF}$.

III. *Solution by* C. W. Merrifield, F.R.S.

That $\mathrm{AF}\,.\,\mathrm{BD}\,.\,\mathrm{CE} = \mathrm{AE}\,.\,\mathrm{BF}\,.\,\mathrm{CD}$, when the lines pass through O, is well known; and since we know that $\mathrm{R} = \dfrac{a}{2\sin \mathrm{A}} = \dfrac{abc}{4\triangle \mathrm{ABC}}$, the second part is equivalent to

$$\frac{2\,\mathrm{AF}\,.\,\mathrm{BD}\,.\,\mathrm{CE}}{abc} = \frac{2\,\mathrm{BF}\,.\,\mathrm{CD}\,.\,\mathrm{AE}}{abc} = \frac{\triangle \mathrm{DEF}}{\triangle \mathrm{ABC}}.$$

Now if the points D, E, F, divide the sides in the ratios $l:m$, $m:n$, $n:l$, as they may do if AD, &c., meet in O, then the point O divides the lines AD, &c., in the ratios $l : m+n$, &c., and we have

$$\frac{\triangle \mathrm{OFE}}{\triangle \mathrm{OBC}} = \frac{\mathrm{OE}\,.\,\mathrm{OF}}{\mathrm{OB}\,.\,\mathrm{OC}} = \frac{m}{n+l}\,.\,\frac{n}{l+m};$$

therefore $$\frac{\triangle \mathrm{OFE}}{\triangle \mathrm{ABC}} = \frac{l}{+n}\,.\,\frac{m}{n+l}\,.\,\frac{n}{l+m} \times \frac{m+n}{l+m+n},$$

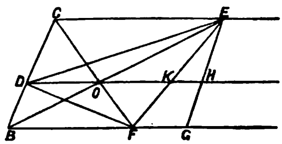
with similar values for the triangles ODE and ODF;

therefore $$\frac{\triangle \mathrm{DEF}}{\triangle \mathrm{ABC}} = \frac{\mathrm{OFE}+\mathrm{OED}+\mathrm{ODF}}{\mathrm{ABC}}$$

$$= \frac{l}{m+n}\,.\,\frac{m}{n+l}\,.\,\frac{n}{l+m}\,.\,\frac{(m+n)+(n+l)+(l+m)}{l+m+n}$$

$$= \frac{2l}{l+m}\,.\,\frac{m}{m+n}\,.\,\frac{n}{n+l} = 2\,\frac{\mathrm{BF}}{\mathrm{AB}}\,.\,\frac{\mathrm{CD}}{\mathrm{BC}}\,.\,\frac{\mathrm{AE}}{\mathrm{CA}}.$$

IV. *Solution by* C. Leudesdorf; H. Murphy; *and others.*



Project A to an infinite distance; then

$$\frac{\mathrm{AF}}{2\mathrm{R}} = \frac{\mathrm{AF}}{\mathrm{AB}}\sin \mathrm{C} = \sin \mathrm{C} \;\ldots\ldots\; (1),$$

Again, if EHG be drawn parallel to BC,

$$\mathrm{KH} : \mathrm{FG} = \mathrm{CD} : \mathrm{CB};$$

therefore $$\mathrm{DK} = \mathrm{CE} - \frac{\mathrm{CD}}{\mathrm{CB}}(\mathrm{CE}-\mathrm{BF})$$

$$= \frac{\mathrm{CE}\,.\,\mathrm{BD}+\mathrm{CD}\,.\,\mathrm{BF}}{\mathrm{CB}} = \frac{2\,\mathrm{CE}\,.\,\mathrm{BD}}{\mathrm{CB}};$$

therefore $\triangle \mathrm{DEF} = \tfrac{1}{2}\mathrm{DK}\,.\,\mathrm{BC}\sin \mathrm{C} = \mathrm{CE}\,.\,\mathrm{BD}\,.\,\sin \mathrm{C}$ .... (2).

From (1) and (2), $2\mathrm{R}\,.\,\triangle \mathrm{DEF} = \mathrm{AF}\,.\,\mathrm{BD}\,.\,\mathrm{CE} = \mathrm{BF}\,.\,\mathrm{CD}\,.\,\mathrm{AE}$.

3430. (Proposed by the Editor.)—Find the equation of the first negative focal pedal of (1) an ellipsoid, and (2) an ellipse.

Solution by Professor Cayley.

1. It is easily seen that if a sphere be drawn, passing through the centre of the given quadric and touching it at any point (x', y', z'), then the point (x, y, z) on the required surface, which corresponds to (x', y', z'), is the extremity of the diameter of this sphere which passes through the centre of the quadric. We thus easily find the expressions

$$x = x'\left(2 - \frac{t}{a^2}\right), \quad y = y'\left(2 - \frac{t}{b^2}\right), \quad z = z'\left(2 - \frac{t}{c^2}\right);$$

where $t = x'^2 + y'^2 + z'^2$.

Solving these equations for x' and y' and z', and substituting in the two equations $xx' + yy' + zz' = x'^2 + y'^2 + z'^2$, $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1$, we get

$$\frac{x^2}{\left(2 - \frac{t}{a^2}\right)} + \frac{y^2}{\left(2 - \frac{t}{b^2}\right)} + \frac{z^2}{\left(2 - \frac{t}{c^2}\right)} = t \text{ .... (1),}$$

$$\frac{x^2}{\left(2 - \frac{t}{a^2}\right)^2} + \frac{y^2}{\left(2 - \frac{t}{b^2}\right)} + \frac{z^2}{\left(2 - \frac{t}{c^2}\right)} = 1 \text{ .... (2).}$$

Since (2) is the differential with respect to t of (1), the result of eliminating t between these two equations is the discriminant of (1). Hence the equation of the required surface is the discriminant of (1) with respect to t. Since (1) is only of the 4th degree, this discriminant is easily formed. If (1) be written in the form

$$At^4 + 4Bt^3 + 6Ct^2 + 4Dt + E = 0,$$

it will be found that A and B do not contain x, y, z, while C, D, E contain them, each in the second degree. Now the discriminant is of the sixth degree in the coefficients, and of the form $A\phi + B^2\psi$ (see Salmon's *Higher Algebra*, § 107); hence it contains x, y, z only in the tenth degree. This is therefore the degree of the required surface.

The section of this derived surface by the principal plane z consists of the discriminant of

$$\frac{x^2}{2 - \frac{t}{a^2}} + \frac{y^2}{2 - \frac{t}{b^2}} = t \text{ .... (3),}$$

(which is of the sixth degree, and is the first negative pedal of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$), together with the conic (taken twice), which is obtained by putting $t = 2c^2$ in (3).

This conic, which is a double curve on the surface, touches the curve of the sixth degree in four points.

2. The formulæ for the conic are quite analogous to those for the ellipsoid, viz., we have

$$x = X\left(2 - \frac{1}{a^2}(X^2 + Y^2)\right), \quad y = Y\left(2 - \frac{1}{b^2}(X^2 + Y^2)\right),$$

leading to the equations $\quad \theta = \dfrac{x^2}{2-\dfrac{\theta}{a^2}} + \dfrac{y^2}{2-\dfrac{\theta}{b^2}}$,

and its derived equation, from which to eliminate θ; viz., the first is the cubic equation $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D})(\theta, 1)^3 = \theta$, where

$$\mathrm{A} = 1, \quad \mathrm{B} = -\tfrac{2}{3}a^2 + b^2, \quad \mathrm{C} = \tfrac{1}{3}a^2x^2 + b^2y^2 + 4a^2b^2, \quad \mathrm{D} = -2a^2b^2(x^2+y^2).$$

So that, equating the discriminant to zero, this is

$$0 = \mathrm{A}^2\nabla = 4(\mathrm{AC}-\mathrm{B}^2)^3 - (3\mathrm{ABC}-\mathrm{A}^2\mathrm{D}-2\mathrm{B}^3)^2.$$

Or finally $\quad (3a^2x^2 + 3b^2y^2 - 4a^4 + 4a^2b^2 - 4b^4)^3$

$$+\left\{9(a^2-2b^2)a^2x^2 + 9(b^2-2a^2)b^2y^2 - 8a^6 + 12a^4b^2 + 12a^2b^4 - b^6\right\}^2 = 0,$$

which is the required equation.

[Solutions by the Editor, Dr. Booth, and Mr. Carr, are given in the *Reprint*, Vol. XVI., pp. 77—83, and Vol. XVII., pp. 92—96, Quest. 3550.]

3431. (Proposed by the Rev. Dr. Booth, F.R.S.)—Find the equation of the first negative central pedal of the ellipse.

Solution by J. L. Mackenzie.

The first negative central pedal of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is the envelope of

$$\alpha x + \beta y = \alpha^2 + \beta^2 \quad \ldots\ldots (1),$$

where α and β are variable parameters satisfying the condition

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1 \quad \ldots\ldots (2).$$

Differentiating (1) and (2), we have

$$x d\alpha + y d\beta - 2(\alpha d\alpha + \beta d\beta) = 0, \qquad \frac{\alpha d\alpha}{a^2} + \frac{\beta d\beta}{b^2} = 0 \ldots\ldots (3, 4).$$

Multiplying (4) by the indeterminate U^2, adding it to (3), and equating to zero the coefficients of the differentials,

$$\frac{\mathrm{U}^2\alpha}{a^2} = 2\alpha - x, \qquad \frac{\mathrm{U}^2\beta}{b^2} = 2\beta - y \ldots\ldots (5, 6).$$

$\alpha(5) + \beta(6)$ gives, along with (2),

$$\mathrm{U}^2\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right) = \mathrm{U}^2 = \alpha(2\alpha - x) + \beta(2\beta - y) = 2(\alpha^2+\beta^2) - (\alpha x + \beta y),$$

whence

$$\mathrm{U}^2 = \alpha^2 + \beta^2 \quad \ldots\ldots (7).$$

Solving (5) and (6) for α and β, we have

$$\alpha = \frac{a^2x}{2a^2 - \mathrm{U}^2}, \quad \beta = \frac{b^2y}{2b^2 - \mathrm{U}^2} \ldots\ldots (8).$$

Substituting in (1) and (2), we have

$$\frac{a^2x^2}{2a^2-\mathrm{U}^2} + \frac{b^2y^2}{2b^2-\mathrm{U}^2} = \mathrm{U}^2, \quad \frac{a^2x^2}{(2a^2-\mathrm{U}^2)^2} + \frac{b^2y^2}{(2b^2-\mathrm{U}^2)^2} = 1 \ldots\ldots (9, 10).$$

We have now to eliminate U^2 between (9) and (10). Since (10) is the differential of (9) with respect to U^2, the resultant of elimination will be the discriminant of (9). Expanding (9) and writing Q^4 for $a^2x^2+b^2y^2+4a^2b^2$, and R^2 for x^2+y^2, as in Dr. BOOTH's investigation,*

$$U^6-2(a^2+b^2)U^4+Q^4U^2-2a^2b^2R^2=0.$$

The discriminant of this cubic in U^2, equated to zero, gives as the equation of the required pedal

$$Q^{12}-(a^2+b^2)^2Q^8-18a^2b^2(a^2+b^2)R^2Q^4+16a^2b^2(a^2+b^2)R^2+27a^4b^4R^2=0.$$

[Other Solutions, by the EDITOR and Mr. WATSON, are given on pp. 83—85 of Vol. XVI. of the *Reprint*.]

4207. (Proposed by A. B. EVANS, M.A.)—Solve the equations

$$x+y+z=12,\quad x^3+y^3+z^3=288,\quad x^9+y^9+z^9=10340352\ \ldots(1,2,3).$$

I. *Solution by* R. TUCKER, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

Let x, y, z be the roots of $d^3-pd^2+qd-r=0$; therefore

$$p=x+y+z=12,\quad q=xy+yz+zx,\quad r=xyz;$$

$$288=x^3+y^3+z^3=(x+y+z)^3-3(x+y+z)(xy+yz+zx)+3xyz$$
$$=1728-369+3r,\ i.e.\ 12q-r=480\ \ldots\ldots\ldots\ldots(\alpha);$$

$$10340352=(288)^3-3(288)(x^3y^3+y^3z^3+z^3x^3)+3r^3.$$

Now $$q^3=(xy+yz+zx)^3=x^3y^3+y^3z^3+z^3x^3+3pqr-3r^2;$$

hence $$x^3y^3+y^3z^3+z^3x^3=q^3-36qr+3r^2,$$

and we get $$288(q^3-36qr+3r^2)-r^3=4515840;$$

that is $$288(q^3-1440r)-r^3=4515840\ \ldots\ldots\ldots\ldots(\beta).$$

Eliminating q we get

$$r^3-288r^2+359424r-16699392=0=(r-48)(r^2-240r+347904),$$

whence $r=48$ and $q=44$. Hence x, y, z will be the roots of

$$d^3-12d^2+44d-48=0=(d-6)(d-4)(d-2),$$

and the following answers are obtained; viz.

$$\begin{array}{l|c|c|c|c|c|c|}x= & 6 & 6 & 4 & 4 & 2 & 2\\ y= & 4 & 2 & 6 & 2 & 4 & 6\\ z= & 2 & 4 & 2 & 6 & 6 & 4\end{array}.$$

The other values are imaginary.

II. *Solution by* ARTEMAS MARTIN.

Put $12=s$, $288=b$, and $10340352=c$; and let

$$xy+xz+yz=p,\quad xyz=q\ \ldots\ldots\ldots\ldots(4,5);$$

then $$x^3+y^3+z^3=s^3-3sp+3q=b\ \ldots\ldots\ldots\ldots(6),$$

$$x^9+y^9+z^9=b^3-4b(p^3+bq-s^3q)+3q^3=c\ \ldots\ldots\ldots\ldots(7).$$

* Se Dr. Booth's *Treatise on some New Geometrical Methods*, p. 143, Art. 145.

Restoring the numbers and reducing, we have

$$12p-q = 480, \quad 288p^3-q^3-414720q = 4515840 \quad \text{......... (8, 9).}$$

From (8), $$p = \frac{q+480}{12} \quad \text{..................................(10);}$$

therefore by (9) $$q^3-288q^2+359424q = 16699392 \quad \text{.................. (11).}$$

Now let $v+96 = q$, and we have $v^3+331776v = -16035840$ ...... (12).

Multiplying (12) by v, and then adding $(167040)^2+2304v^2$ to each side,

$$v^4+334080v^2+(167040)^2 = (167040)^2-16035840v+2304v^2 \quad \text{...... (13).}$$

Extracting square root of (13), we have

$$v^2+167040 = 167040-48v \quad \text{......................... (14).}$$

Whence $v = -48$; therefore $q = 48 = xyz$ .... (15).

Substituting in (10), we find $p = 44 = xy+xz+yz$ .... (16).

From (1), (16), (15) we have, by the Theory of Equations,

$$x^3-12x^2+44x-48 = 0 \quad \text{......................... (17),}$$

the three roots of which cubic are the values of x, y, z required.

Multiplying (17) by x it may be written

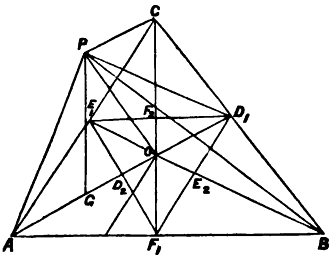
$$(x^2-6x)^2+8(x^2-6x) = 0 \quad \text{......................... (18).}$$

Whence $x^2-6x = 0$, which gives $x=6$. Also $x^2-6x+8 = 0$, from which we obtain $x = 4$ or 2.

4215. (Proposed by S. WATSON.)—Let ABC be a triangle; O its centroid; D_1, E_1, F_1 the middle points of BC, CA, AB; D_2, E_2, F_2 the middle points of E_1F_1, F_1D_1, D_1E_1; and so on. Then, if P be any point in the plane of the triangle, prove that

$$PD_n^2 + PE_n^2 + PF_n^2 = 3PO^2 + \frac{1}{3 \cdot 4^n}(AB^2+BC^2+CA^2).$$

I. *Solution by the* Rev. J. L. KITCHIN, M.A.



Taking AB, AC as axes, the coordinates of D_1, E_1, F_1, and D_2, E_2, F_2 are $(\frac{1}{2}c, \frac{1}{2}b)$, $(0, \frac{1}{2}b)$, $(\frac{1}{2}c, 0)$, and $(\frac{1}{4}c, \frac{1}{4}b)$, $(\frac{1}{2}c, \frac{1}{4}b)$, $(\frac{1}{4}c, \frac{1}{2}b)$; also the coordinates of O are $(\frac{1}{3}c, \frac{1}{3}b)$; hence, putting (h, k) for the coordinates of P, we have

$$PD_2^2 = \left(h-\frac{c}{4}\right)^2+\left(k-\frac{b}{4}\right)^2 + 2\left(h-\frac{c}{4}\right)\left(k-\frac{b}{4}\right)\cos A,$$

$PE_2^2 = \&c.$, $PF_2^2 = \&c.$;

therefore

$$PD_2^2+PE_2^2+PF_2^2=2\left(h-\frac{c}{4}\right)^2+2\left(k-\frac{b}{4}\right)^2+\left(h-\frac{c}{2}\right)^2+\left(k-\frac{b}{2}\right)^2$$

$$+2\left\{\left(h-\frac{c}{4}\right)\left(k-\frac{b}{4}\right)+\left(h-\frac{c}{2}\right)\left(k-\frac{b}{4}\right)+\left(h-\frac{c}{4}\right)\left(k-\frac{b}{2}\right)\right\}\cos A$$

$$=2\left(\overline{h-\frac{c}{3}}+\frac{c}{12}\right)^2+2\left(\overline{k-\frac{b}{3}}+\frac{b}{12}\right)^2+\left(\overline{h-\frac{c}{3}}-\frac{c}{6}\right)^2+\left(\overline{k-\frac{b}{3}}-\frac{b}{6}\right)^2$$

$$+2\left\{\left(\overline{h-\frac{c}{3}}+\frac{c}{12}\right)\left(\overline{k-\frac{b}{3}}+\frac{b}{12}\right)+\left(\overline{h-\frac{c}{3}}-\frac{c}{6}\right)\left(\overline{k-\frac{b}{3}}+\frac{b}{12}\right)\right.$$

$$\left.+\left(\overline{h-\frac{c}{3}}+\frac{c}{12}\right)\left(\overline{k-\frac{b}{3}}-\frac{b}{6}\right)\right\}\cos A$$

$$=2\left(h_1+\frac{c}{12}\right)^2+2\left(k_1+\frac{b}{12}\right)^2+\left(h_1-\frac{c}{6}\right)^2+\left(k_1-\frac{b}{6}\right)^2$$

$$+2\left\{\left(h_1+\frac{c}{12}\right)\left(k_1+\frac{b}{12}\right)+\left(h_1-\frac{c}{6}\right)\left(k_1+\frac{b}{12}\right)\right.$$

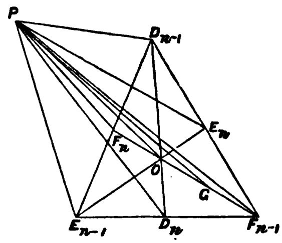
$$\left.+\left(h_1+\frac{c}{12}\right)\left(k_1-\frac{b}{6}\right)\right\}\cos A.$$

$$=3PO^2+\frac{1}{48}(c^2+b^2+a^2)=3PO^2+\frac{1}{3\,.\,4^2}(a^2+b^2+c^2).$$

The same may be proved for any number of triangles drawn as in the question; hence we obtain, finally,

$$PD_n^2+PE_n^2+PF_n^2=3PO^2+\frac{1}{3\,.\,4^n}\left\{a^2+b^2+c^2\right\}.$$

II. *Solution by* Asher B. Evans, M.A.



It is evident, from the construction of the triangles $D_1E_1F_1$, $D_2E_2F_2$, ... $D_nE_nF_n$, that they are similar to ABC, and that O is their common centroid. It is easily shown that the homologous sides of these triangles are in geometrical progression, and that $D_{n-1}E_{n-1}=\frac{1}{2^{n-1}}(AB)$.

$$E_{n-1}F_{n-1}=\frac{1}{2^{n-1}}(BC),$$

$$F_{n-1}D_{n-1}=\frac{1}{2^{n-1}}(CA);$$

from which we obtain

$$D_{n-1}E_{n-1}^2+E_{n-1}F_{n-1}^2+F_{n-1}D_{n-1}^2=\frac{1}{4^{n-1}}(AB^2+BC^2+CA^2)\ \ldots\ldots\ (1).$$

If $D_{n-1}E_{n-1}F_{n-1}$ be a triangle, D_n, E_n, F_n the middle points of $E_{n-1}F_{n-1}$, $F_{n-1}D_{n-1}$, $D_{n-1}E_{n-1}$, O its centroid and P any point in its plane, it is easily shown that

$$PD_n^2 + PE_n^2 + PF_n^2 = 3PO^2 + \frac{1}{12}(D_{n-1}E_{n-1}^2 + E_{n-1}F_{n-1}^2 + F_{n-1}D_{n-1}^2) \qquad \ldots\ldots (2).$$

From (1) and (2) we deduce immediately

$$PD_n^2 + PE_n^2 + PF_n^2 = 3PO^2 + \frac{1}{3 \cdot 4^n}(AB^2 + BC^2 + CA^2).$$

[Equation (2) may be readily obtained by bisecting OF_{n-1} in G and applying to the various triangles of the figure the following theorem:— The squares on two sides of a triangle are equal to twice the square on half the third side, together with twice the square on the medial line to that side from the opposite angle.]

III. *Solution by the* PROPOSER.

Since O is at the intersection of AD_1, BE_1, CF_1, and those lines bisect E_1F_1, F_1D_1, D_1E_1 respectively, it follows by this and symmetry that O is the common centroid of all the triangles ABC, $D_1E_1F_1$, $D_2E_2F_2$, &c. Bisect AO in G, and draw PA, PB, PC, PD_1, PO, PG. Then, since $AO = 2OD_1$, therefore $AG = GO = OD_1 = \frac{1}{3}AD_1$. Now, by a well known property $\quad AB^2 + AC^2 = 2(AD_1^2 + BD_1^2) - 18\,OD_1^2 + 2\,BD_1^2$,

therefore $\qquad AB^2 + BC^2 + CA^2 = 18\,OD_1^2 + 6\,BD_1^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1).$

Again, by the same property, we have

$$PA^2 + PO^2 = 2(OG^2 + PG^2), \quad PB^2 + PC^2 = 2(PD_1^2 + BD_1^2),$$
$$2(PD_1^2 + PG^2) = 4(PO^2 + OD_1^2).$$

Hence, by addition, and taking away what is common to both sides,

$$PA^2 + PB^2 + PC^2 = 3\,PO^2 + 6\,OD_1^2 + 2\,BD_1^2,$$

which by (1) $\qquad = 3\,PO^2 + \frac{1}{3}(AB^2 + BC^2 + CA^2).$

Similarly $\quad PD_1^2 + PE_1^2 + PF_1^2 = 3\,PO^2 + \frac{1}{3}(D_1E_1^2 + E_1F_1^2 + F_1D_1^2)$

$$= 3\,PO^2 + \frac{1}{3 \cdot 4}(AB^2 + BC^2 + CA^2).$$

Hence, generally, $PD_n^2 + PE_n^2 + PF_n^2 = 3\,PO^2 + \frac{1}{3 \cdot 4^n}(AB^2 + BC^2 + CA^2).$

IV. *Solution by* N'IMPORTE.

Draw PA, PB, PC, PD_1, PO, D_1E_1, E_1F_1, F_1D_1. Now it is well known that AD_1, BE_1, CF_1 pass through O, and that $OD_1 = \frac{1}{3}AD$ and $AO = \frac{2}{3}AD$. Also, it is proved in *Simpson's Select Exercises* and elsewhere, that if a side BC of a triangle ABC be so divided at D_1, that $m \cdot BD_1 = n \cdot D_1C$,

then $\qquad m \cdot AB^2 + n \cdot AC^2 = (m + n)(AD_1^2 + BD_1 \cdot D_1C).$

Hence, $\qquad PA^2 + 2PD_1^2 = 3\,PO^2 + 3\,AO \cdot OD_1,$

and $\qquad PB^2 + PC^2 = 2\,PD_1^2 + 2\,BD_1^2.$

Therefore, adding $PA^2+PB^2+PC^2 = 3PO + 3AO.OD_1+2BD_1^2,$

$$= 3PO^2+\tfrac{2}{3}AD_1^2+2BD_1^2 = 3PO^2+\tfrac{4}{3}BD_1^2+\tfrac{1}{3}(2BD_1^2+2AD_1^2)$$
$$= 3PO^2+\tfrac{4}{3}BD_1^2+\tfrac{1}{3}(AB^2+AC^2) = 3PO^2+\tfrac{1}{3}(AB^2+BC^2+AC^2).$$

Hence, symmetrically, we have

$$PD_n^2 + PE_n^2 + PF_n^2 - 3PO^2+\tfrac{1}{3}(D_nE_n^2+E_nF_n^2+F_nD_n^2) \text{ ...... (1).}$$

But $D_1E_1 = \tfrac{1}{2}AB$, therefore $D_1E_1^2 = \tfrac{1}{4}AB^2$;

$D_2E_2 = \tfrac{1}{4}AB$, therefore $D_2E_2^2 = \dfrac{1}{4^2}AB^2$;

$D_3E_3 = \tfrac{1}{8}AB$, therefore $D_3E_3^2 = \tfrac{1}{64}AB^2 = \dfrac{1}{4^3}AB^2$;

hence obviously $D_nE_n^2 = \dfrac{1}{4^n}AB^2$;

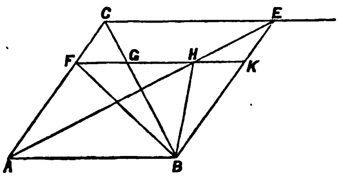
similarly, $E_nF_n^2 = \dfrac{1}{4^n}BC^2$, and $F_nD_n^2 = \dfrac{1}{4^n}CA^2$.

Consequently, by substitution in (1), we get

$$PD_n^2 + PE_n^2 + PF_n^2 = 3PO^2 + \frac{1}{3.4^n}(AB^2+BC^2+CA^2)$$

4170. (Proposed by Professor CROFTON, F.R.S.)—If a triangle be deformed (geometrically) into any other on the same base and between the same parallels, (each point of its surface having moved parallel to the base, and there being no relative motion among the points on any parallel,) show that there will be no superficial dilatation or contraction in any part of the surface.

Solution by the Rev. J. L. KITCHIN, M.A.



Draw any line FGHK. Then since there is no relative motion of the points on any line parallel to AB, FG = HK; and this is true for any position; hence there is no contraction or dilatation parallel to AB.

There is obviously none perpendicular to AB, therefore there is none whatever.

The non-relative motion is a necessity; for if FG were not equal to HK, the triangles would not be equal; which is absurd.

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